

LAPLACE AND POISSON EQUATIONS - UNIQUENESS OF SOLUTIONS

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We've seen that the electric field obeys Gauss's law, which in differential form is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (1)$$

The field can also be written as the gradient of a potential function, so we get

$$\mathbf{E} = -\nabla V \quad (2)$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad (3)$$

The last equation is a partial differential equation (PDE) known as Poisson's equation, and its solution gives the potential for a given charge distribution.

In regions where there is no charge, $\rho = 0$ and Poisson's equation becomes Laplace's equation:

$$\nabla^2 V = 0 \quad (4)$$

Two standard problems in PDE theory are proofs that for a particular PDE a solution (a) exists and (b) is unique. (Somewhat bizarrely, some mathematicians are content with knowing these two facts without actually having a method for *finding* the solution, but we'll leave that for now.)

In electrostatics, we are faced with proving these things for Poisson's and Laplace's equations. Actually, the proof of the existence of solutions requires some heavy-duty mathematics and we won't go into it here. Usually we can be confident that a solution exists provided we know that we've specified the problem correctly in physical terms.

What constitutes a correct specification of such an electrostatic problem? We need to specify the charge distribution ρ and also some boundary conditions. PDEs can have a variety of boundary conditions. Two such conditions that are important enough to have been named after someone are as follows.

1. The Dirichlet problem involves finding the solution of a PDE in which the values of the solution function V are specified on the boundaries. In electrostatics, this means specifying the potential on all the boundary surfaces.

2. The Neumann problem requires finding the solution when its normal derivative $\frac{\partial V}{\partial n}$ (that is, the derivative normal to a surface) is specified on all boundaries. In electrostatics, this means specifying the component of the electric field normal to the boundary surfaces (since $E_{\perp} = \mathbf{E} \cdot \hat{\mathbf{n}} = -\nabla V \cdot \hat{\mathbf{n}} = -\frac{\partial V}{\partial n}$, where $\hat{\mathbf{n}}$ is the unit normal to the surface).

Needless to say, a general PDE problem can involve combinations of Dirichlet and Neumann conditions.

Actually finding solutions satisfying these conditions can be challenging, and in some (probably most) cases, only numerical solutions can be found. However, before we get to techniques for solving Poisson's and Laplace's equations, we need to establish the very useful fact that any solution to a given problem is unique.

First, we'll consider the Dirichlet problem, and examine the case where we have specified the charge distribution ρ and also the values of the potential V on some set of boundary surfaces. We want to show that any solution in such a case is unique. To do this, we suppose to the contrary that there are, in fact, two solutions V_1 and V_2 , and then examine the difference $V_3 \equiv V_1 - V_2$. First, from Poisson's equation, we know that

$$\nabla^2 V_1 = \nabla^2 V_2 = -\frac{\rho}{\epsilon_0} \quad (5)$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 \quad (6)$$

$$= 0 \quad (7)$$

Thus the difference between these two solutions satisfies Laplace's equation. Now, since we've specified the boundary values of the potential, we know that $V_1 = V_2$ everywhere on the boundary, so $V_3 = 0$ on all boundaries. Finally, we know that Laplace's equation has no maxima or minima except on the boundaries, so that must mean that both the maximum and minimum values of V_3 are zero, which means that $V_3 = 0$ everywhere, so $V_1 = V_2$. So any solution to the Dirichlet problem with Poisson's (and hence, Laplace's) equation is unique.

How about the Neumann case? Since we're dealing with normal derivatives of the potential, it is more natural to work with the electric field than with the potential, at least at first. We'll again start by assuming that there are two separate fields that satisfy the boundary conditions. From Gauss's law we know that

$$\nabla \cdot \mathbf{E}_1 = \nabla \cdot \mathbf{E}_2 = \frac{\rho}{\epsilon_0} \quad (8)$$

The surface integral of the electric field is

$$\oint \mathbf{E}_1 \cdot d\mathbf{a} = - \oint \frac{\partial V_1}{\partial n} da \quad (9)$$

Since we have specified the normal derivative at the boundaries, we therefore have that $\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$ over all boundaries, so on these boundaries we must have

$$\mathbf{E}_1 \cdot d\mathbf{a} = \mathbf{E}_2 \cdot d\mathbf{a} \quad (10)$$

We can now consider the difference $\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2$ between these two fields, and conclude that

$$\nabla \cdot \mathbf{E}_3 = 0 \quad (11)$$

$$\mathbf{E}_3 \cdot d\mathbf{a} = 0 \quad (12)$$

At this point we can use an identity from vector calculus. For any scalar field C and vector field \mathbf{A} , we have

$$\nabla \cdot (C\mathbf{A}) = C\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla C \quad (13)$$

If we introduce the difference in potentials as before: $V_3 \equiv V_1 - V_2$ and apply this result to V_3 and \mathbf{E}_3 , we get

$$\nabla \cdot (V_3\mathbf{E}_3) = V_3\nabla \cdot \mathbf{E}_3 + \mathbf{E}_3 \cdot \nabla V_3 \quad (14)$$

Since $\mathbf{E}_3 = -\nabla V_3$ and we know from above that $\nabla \cdot \mathbf{E}_3 = 0$ we get

$$\nabla \cdot (V_3\mathbf{E}_3) = -E_3^2 \quad (15)$$

Applying the divergence theorem, we get

$$\int \nabla \cdot (V_3\mathbf{E}_3) d^3\mathbf{r} = \oint V_3\mathbf{E}_3 \cdot d\mathbf{a} \quad (16)$$

$$= - \int E_3^2 d^3\mathbf{r} \quad (17)$$

The LHS integral in the first line and the integral in the second line are both volume integrals over all space bounded by the boundaries, while the RHS integral in the first line is a surface integral. We see from above that the RHS of the first line is zero because of the Neumann boundary conditions, so from the second line we get

$$\int E_3^2 d^3\mathbf{r} = 0 \quad (18)$$

Since the integrand is the square of a real function, the integrand is non-negative everywhere, so it must be zero, thus $\mathbf{E}_1 = \mathbf{E}_2$. Thus for Neumann conditions, the field is unique.

Note that this last proof also demonstrates that if we mix the boundary conditions, then the field is still unique. If we apply Dirichlet conditions on some surfaces, then on those surfaces $V_3 = V_1 - V_2 = 0$ and the surface integral above is still zero over those boundaries.

Not surprisingly, if we're dealing with derivatives of V , we can't prove that the potential is unique, since we could add a constant to the potential and still get the same derivative. Thus for Neumann conditions, we can say only that the field \mathbf{E} is unique, but not the potential V .

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