LAPLACE'S EQUATION - FOURIER SERIES

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Here are a few examples of calculating the Fourier coefficients in the series solution to Laplace's equation for some special cases.

Example 1. Consider the infinite slot problem with the boundary at $x=0$ consisting of a conducting strip with a constant potential of V_0. In this case we get

$$c_n = \frac{2V_0}{a}\int_0^a \sin\frac{n\pi y}{a}dy \tag{1}$$

$$= \frac{2V_0}{n\pi}(1-\cos n\pi) \tag{2}$$

The coefficients are thus zero for even n and $4V_0/\pi$ for odd n:

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi} & n \text{ odd} \end{cases} \tag{3}$$

The potential is thus

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{e^{-n\pi x/a}}{n}\sin\frac{n\pi y}{a} \tag{4}$$

Example 2. Now suppose the boundary at $x=0$ consists of two conducting strips, insulated from each other and from the infinite sheets. The first strip, from $y=0$ to $y=a/2$ has a constant potential V_0 while the other strip, from $y=a/2$ to $y=a$ is held at potential $-V_0$.

Here, the coefficients c_n are given by

$$c_n = \frac{2V_0}{a}\left[\int_0^{a/2}\sin\frac{n\pi y}{a}dy - \int_{a/2}^{a}\sin\frac{n\pi y}{a}dy\right] \tag{5}$$

$$= \frac{2V_0}{n\pi}\left[1-2\cos\frac{n\pi}{2}+\cos n\pi\right] \tag{6}$$

If n is odd, this comes out to zero. If n is even, there are two cases. First, if $n=2,6,10,\ldots$ the term in brackets is 4. If $n=4,8,12,\ldots$ the term in brackets is zero. Thus we get

$$c_n=\begin{cases}0 & n\text{ odd}\\ \frac{8V_0}{n\pi} & n=2,6,10\ldots\\ 0 & n=4,8,12\ldots\end{cases} \tag{7}$$

Thus the potential is

$$V(x,y)=\frac{8V_0}{\pi}\sum_{n=2,6,10\ldots}^{\infty}\frac{e^{-n\pi x/a}}{n}\sin\frac{n\pi y}{a} \tag{8}$$

$$=\frac{8V_0}{\pi}\sum_{n=0}^{\infty}\frac{e^{-(4n+2)\pi x/a}}{4n+2}\sin\frac{(4n+2)\pi y}{a} \tag{9}$$

where in the last line we've changed the index of summation since the non-zero terms in the first sum are just those with $n=4m+2$ starting at $m=0$.

Example 3. The infinite slot with the strip at $x=0$ held at potential V_0 has the solution

$$V(x,y)=\frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{e^{-n\pi x/a}}{n}\sin\frac{n\pi y}{a} \tag{10}$$

For a conductor, the surface charge density can be found from the derivative taken normal to the surface:

$$\sigma=-\epsilon_0\left.\frac{\partial V}{\partial n}\right|_{x=0} \tag{11}$$

In this case, the normal to the surface is the x direction, so we get

$$\sigma=-\epsilon_0\frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\left(-\frac{n\pi}{a}\right)\frac{e^{-n\pi x/a}}{n}\sin\left.\frac{n\pi y}{a}\right|_{x=0} \tag{12}$$

$$=\epsilon_0\frac{4V_0}{a}\sum_{n=1,3,5,\ldots}^{\infty}\sin\frac{n\pi y}{a} \tag{13}$$

This looks fine except for the problem that the series doesn't converge. Consider $y=a/2$. The series is then a sum of an alternating sequence of $+1$ and -1. The original series for the potential does converge at $x=0$ due to the n in the denominator. Not sure what the solution to this is.

Example 4. We have an infinite rectangular pipe extending to infinity in both directions, lying parallel to the z axis. The four sides of the pipe are as follows.

At $y = 0$ and $y = a$ the potential is held at $V = 0$. At $x = 0$ the potential is also $V = 0$, but at $x = b$ it is some arbitrary function of y: $V = V_0(y)$. We can use separation of variables and Fourier series to find the potential everywhere inside the pipe.

The general solution from separation of variables gives us

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky) \tag{14}$$

for constants A, B, C, D. In this case we could choose to swap x and y in the solution, since neither x nor y goes to infinity so there's no requirement for either term to vanish at infinity. However, with the given boundary conditions, the current choice makes things easier (though feel free to try it the other way round if you like; that is, try a solution of form $V(x,y) = \left(Ae^{ky} + Be^{-ky}\right)(C\sin kx + D\cos kx)$and see how far you get).

The boundary conditions are

$$V = \begin{cases} 0 & y = 0 \\ 0 & y = a \\ 0 & x = 0 \\ V_0(y) & x = b \end{cases} \tag{15}$$

The first condition gives

$$D = 0 \tag{16}$$

The second gives

$$k = \frac{n\pi}{a} \tag{17}$$

for $n = 1, 2, 3, \ldots$

The third gives

$$A + B = 0 \tag{18}$$

$$A = -B \tag{19}$$

We therefore get, for a particular choice of n:

$$V_n(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin\frac{n\pi}{a}y \quad (20)$$
$$= 2AC\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a} \quad (21)$$
$$\equiv c_n\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a} \quad (22)$$

where in the last line we've merged the constant $2AC$ into the single constant c_n.

As usual, we can now form the general solution as a series of V_n terms:

$$V(x,y) = \sum_{n=1}^{\infty} c_n\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a} \quad (23)$$

The coefficients c_n can be found from the fourth boundary condition above, by multiplying both sides by $\sin\frac{m\pi y}{a}$ and integrating.

$$\int_0^a V_0(y)\sin\frac{m\pi y}{a}dy = \sum_{n=1}^{\infty} c_n\sinh\frac{n\pi b}{a}\int_0^a \sin\frac{m\pi y}{a}\sin\frac{n\pi y}{a}dy \quad (24)$$
$$= \frac{a}{2}c_m\sinh\frac{m\pi b}{a} \quad (25)$$

Reverting to using n as the subscript on the coefficients, we get

$$c_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a V_0(y)\sin\frac{n\pi y}{a}dy \quad (26)$$

We can't go any further without specifying $V_0(y)$.

In the special case where $V_0(y) = V_0$ =constant, we can work out the integral on the right and get

$$c_n = \begin{cases} 0 & n = 2,4,6,\ldots \\ \frac{4V_0}{n\pi\sinh\frac{n\pi b}{a}} & n = 1,3,5,\ldots \end{cases} \quad (27)$$

In this case, the general solution is

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{\sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}}{n\sinh\frac{n\pi b}{a}} \quad (28)$$

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