LAPLACE'S EQUATION - CYLINDRICAL SHELL

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As another example of applying the solution to Laplace's equation in cylindrical coordinates, we consider the following problem. We are given a cylindrical non-conducting shell or radius R carrying a charge density of

$$\sigma(\phi) = k\sin 5\phi \tag{1}$$

We wish to find the potential outside and inside the cylinder.

We start with the general solution:

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n - \frac{C_n}{r^n}\right)\sin n\phi + \sum_{n=1}^{\infty}\left(B_n r^n + \frac{D_n}{r^n}\right)\cos n\phi \tag{2}$$

Since we are given the surface charge, we can use a similar procedure to that used in solving problems in spherical coordinates. We can start by finding the general form of the potential in the two regions. Outside the shell, to keep V finite, we eliminate the $\ln r$ and r^n terms. (In fairness, it has to be pointed out that the potential of an infinite charged wire *does* go as $\ln r$ so in that case it seems acceptable for V to be infinite at large distances. Because the solution to Laplace's equation with a given set of boundary conditions is unique, though, the fact that we can find a solution in this problem in which V remains finite at all locations must mean that it *is* the actual solution.)

Thus, outside the shell, we have

$$V_{\text{out}} = B_{\text{out}} + \sum_{n=1}^{\infty}\left[\frac{D_n}{r^n}\cos n\phi - \frac{C_n}{r^n}\sin n\phi\right] \tag{3}$$

Inside the shell, we eliminate the terms in $\ln r$ and $1/r^n$ to prevent an infinity at $r=0$. We have

$$V_{\text{in}} = B_{\text{in}} + \sum_{n=1}^{\infty}\left[A_n r^n \sin n\phi + B_n r^n \cos n\phi\right] \tag{4}$$

Since the potential is continuous over a surface charge, we must have $V_{\text{out}}(R) = V_{\text{in}}(R)$, so we get

$$B_{\text{out}}+\sum_{n=1}^{\infty}\left[\frac{D_n}{R^n}\cos n\phi-\frac{C_n}{R^n}\sin n\phi\right]=B_{\text{in}}+\sum_{n=1}^{\infty}\left[A_nR^n\sin n\phi+B_nR^n\cos n\phi\right] \tag{5}$$

Equating coefficients of the sine and cosine, we get

$$B_{\text{out}} = B_{\text{in}} \tag{6}$$
$$C_n = -A_nR^{2n} \tag{7}$$
$$D_n = B_nR^{2n} \tag{8}$$

The outward derivative of the potential is discontinuous across a surface charge, and we have

$$\left.\frac{\partial V}{\partial r}\right|_{out}-\left.\frac{\partial V}{\partial r}\right|_{in}=-\frac{\sigma}{\epsilon_0} \tag{9}$$

Plugging in the formulas for V_{out} and V_{in}, we get

$$\sum_{n=1}^{\infty}\left[\frac{-nR^{2n}A_n}{R^{n+1}}-nR^{n-1}A_n\right]\sin n\phi+\sum_{n=1}^{\infty}\left[\frac{-nR^{2n}B_n}{R^{n+1}}-nR^{n-1}B_n\right]\cos n\phi=-\frac{k}{\epsilon_0}\sin 5\phi \tag{10}$$

Equating coefficients of sine and cosine, we see that the only non-zero term on the left must be the $\sin 5\phi$ term, so we have

$$10A_5R^4 = \frac{k}{\epsilon_0} \tag{11}$$
$$A_5 = \frac{k}{10\epsilon_0R^4} \tag{12}$$
$$C_5 = -\frac{k}{10\epsilon_0}R^6 \tag{13}$$
$$A_n = C_n = 0 \quad (n \neq 5) \tag{14}$$
$$B_n = D_n = 0 \quad (all\ n) \tag{15}$$

We are left with the constants $B_{\text{out}} = B_{\text{in}}$ and might as well take them to be zero, since an arbitrary constant doesn't affect the potential. We therefore get

$$V_{\text{out}}(r,\phi) = \frac{kR^6}{10\epsilon_0 r^5}\sin 5\phi \tag{16}$$

$$V_{\text{in}}(r,\phi) = \frac{kr^5}{10\epsilon_0 R^4}\sin 5\phi \tag{17}$$

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