LAPLACE'S EQUATION - SEPARATION OF VARIABLES

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Laplace's equation governs the electric potential in regions where there is no charge. Its form is

$$\nabla^2 V = 0 \tag{1}$$

We've seen that, for a particular set of boundary conditions, solutions to Laplace's equation are unique. That fact can be used to invent the method of images, in which a complex problem can be solved by inventing a simpler problem that has the same boundary conditions.

However, the method of images works only in a few special (and fairly contrived) situations. In the more general case, we need a way of solving Laplace's equation directly.

A method which we have already met in quantum mechanics when solving Schrödinger's equation is that of *separation of variables*. In general, the potential is a function of all three spatial coordinates: $V = V(x,y,z)$. We try to find a solution by assuming that V is a product of three functions, each of which is a function of only one spatial coordinate:

$$V(x,y,z) = X(x)Y(y)Z(z) \tag{2}$$

Substituting this into Laplace's equation, we get

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0 \tag{3}$$

We can divide through by XYZ to get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{4}$$

The key point in this equation is that each term in the sum is a function of only one of the three independent variables x, y and z. The fact that these variables are *independent* is important, for it means that the only way this equation can be satisfied is if each term in the sum is a constant. Suppose this wasn't true; for example, suppose the first term in the sum was some function $f(x)$ that actually does vary with x. Then we could hold y and z

constant and vary x, causing this first term to vary. In this case we cannot satisfy the overall equation, since if we found some value of x for which the sum of the three terms was zero, changing x would change the first term but not the other two, so the overall sum would no longer be zero.

Thus we can say that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \tag{5}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2 \tag{6}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = C_3 \tag{7}$$

where the three constants satisfy

$$C_1 + C_2 + C_3 = 0 \tag{8}$$

Equations of this form have one of two types of solution (well, three, if we consider the constant to be zero, but that's not usually very interesting), depending on whether the constant is positive or negative. For example, if $C_1 > 0$, we can write it as $C_1 = k^2$ and the solution has the form

$$X(x) = Ae^{kx} + Be^{-kx} \tag{9}$$

for some constants A and B.

If $C_1 < 0$, we can write it as $C_1 = -k^2$ and the solution has the form

$$X(x) = D\sin kx + E\cos kx \tag{10}$$

for some constants D and E. The constants in each case must be determined from the boundary conditions. Similar solutions exist for $Y(y)$ and $Z(z)$.

Now you might be wondering whether the assumption that the potential is the product of three separate functions is valid. After all, it does seem to be a rather severe restriction on the solution. It's easiest to see whether this assumption is valid by considering a particular example.

Example 1. The key consideration in any Laplace problem is the specification of the boundary conditions. As a first example, suppose we have the following setup. We have two semi-infinite conducting plates that lie parallel to the xz plane, with their edges in the yz plane, parallel to the z axis (that is, at $x = 0$). One plate is at $y = 0$ (so its edge is the z axis) and the other is at $y = a$ (so its edge is the line $x = 0$, $y = a$, parallel to the z axis). Both plates are grounded, so their potential is constant at $V = 0$.

The strip between the plates at $x = 0$ is filled with another substance (not a conductor, so the potential can vary across it) that is insulated from the two plates, and its potential is some function $V_0(y)$. Solve Laplace's equation to find the potential between the plates.

It's important to note what the boundary conditions are here. The two plates are held at $V = 0$ so provide boundary conditions at $y = 0$ and $y = a$:

$$V(x,0,z) = V(x,a,z) = 0 \tag{11}$$

The strip at $x = 0$ provides another boundary condition

$$V(0,y,z) = V_0(y) \tag{12}$$

Finally, we can impose the condition that the potential drops to zero as we get infinitely far from the strip at $x = 0$ so we have

$$V(\infty,y,z) = 0 \tag{13}$$

The first thing to notice is that none of these boundary conditions depends on z, so we can take $Z(z) =$ constant so that $C_3 = 0$ above. This means that the problem effectively reduces to a two-dimensional problem with the condition

$$C_1 + C_2 = 0 \tag{14}$$

Now we must make a choice as to which of the constants is positive and which is negative. Suppose we chose $C_1 = -k^2 < 0$. Then we would get

$$X(x) = D\sin kx + E\cos kx \tag{15}$$

Looking at the boundary conditions above, we see as $x \to \infty$ we need $X(x) \to 0$. But since $X(x)$ is the sum of two oscillating functions, this can't happen unless $D = E = 0$ or $X = 0$, which isn't a valid solution since that would mean that $V(x,y,z) = 0$ everywhere, and that violates the condition at $x = 0$.

So we can try the other choice: $C_1 = k^2 > 0$. This gives

$$X(x) = Ae^{kx} + Be^{-kx} \tag{16}$$

Now as $x \to \infty$ the negative exponential term drops to zero, so we need only require that $A = 0$ and we get

$$X(x) = Be^{-kx} \tag{17}$$

From this choice, we know that $C_2 = -C_1 = -k^2$ and

$$Y(y) = D\sin ky + E\cos ky \tag{18}$$

From the condition $V = 0$ when $y = 0$ we get

$$E = 0 \tag{19}$$

Finally, from $V = 0$ when $y = a$ we get

$$D \sin ka = 0 \tag{20}$$

$$k = \frac{n\pi}{a} \tag{21}$$

where n is a *positive* integer. It must be non-zero, since $n = 0$ again gives us $V = 0$ everywhere. It must not be negative, since that would give us a negative k which would give the wrong behaviour for $X(x)$.

So our solution so far is

$$V(x,y,z) = BDe^{-n\pi x/a} \sin\left(\frac{n\pi}{a} y\right) \tag{22}$$

At this stage, you might think we've solved ourself into a corner, since we haven't used the final boundary condition which is that $V(0,y,z) = V_0(y)$. From our solution so far, we have

$$V(0,y,z) = BD \sin\left(\frac{n\pi}{a} y\right) \tag{23}$$

so unless we choose $V_0(y)$ to be one of those sine functions, we're stuffed. Does this mean that the separation of variables method doesn't work here?

Not quite. The crucial point is that Laplace's equation is linear (the derivatives occur to the first power only), so any number of separate solutions can be added together to give another solution. That is, if V_1 and V_2 are solutions, then so is $V_1 + V_2$. The separation of variables method has actually given us an infinite number of solutions (one for each value of $n = 1, 2, 3, \ldots$) so we can create yet more solutions by adding together any combination of these individual solutions. In particular, we can say

$$V(x,y,z) = \sum_{n=1}^{\infty} c_n e^{-n\pi x/a} \sin\left(\frac{n\pi}{a} y\right) \tag{24}$$

for some choice of coefficients c_n. (Here, we've simply combined the two constants B and D for each value of n to give the constant c_n.)

How can we find these coefficients? In general, this can be fairly tricky, but for certain boundary conditions, it turns out to be fairly straightforward. For the boundary condition we have here, $V(0,y,z) = V_0(y)$, things aren't too bad. We have

$$V(0,y,z) = V_0(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a} y\right) \tag{25}$$

Some readers might recognize this as a Fourier series, and there is a clever technique that can be used to find the c_n in such a case. We multiply through by $\sin\left(\frac{m\pi y}{a}\right)$ and integrate from 0 to a:

$$\int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0(y)dy = \sum_{n=1}^{\infty} c_n \int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi}{a}y\right) dy \tag{26}$$

The integrals in the sum on the right are fairly straightforward, and we get

$$\int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi}{a}y\right) dy = \begin{cases} 0 \text{ if } m \neq n \\ \frac{a}{2} \text{ if } m = n \end{cases} \tag{27}$$

That is

$$c_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) V_0(y)dy \tag{28}$$

As usual for physicists, the problem of proving that a Fourier series exists and converges for any given function is left to the mathematicians, but for pretty well any function $V_0(y)$ of physical relevance, this technique works. Although the example here has a clean solution, many other problems do not. If the boundaries are of some exotic shape, then it becomes impossible to specify things in such a way that we have a clean Fourier series to work with. As usual in such cases, we need to resort to numerical solution of Laplace's equation, and for that we need a computer.