LAPLACE'S EQUATION IN CYLINDRICAL COORDINATES

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We can use the separation of variables technique to solve Laplace's equation in cylindrical coordinates, in the special case where the potential does not depend on the axial coordinate z. In general, Laplace's equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

We let $V(r,\phi)=R(r)\Phi(\phi)$ and then multiply through by r^2 and divide through by V:

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}=0 \tag{2}$$

Since each term depends only on a separate independent variable, each term must be a constant. We'll consider first the special case where this constant is zero. In this case, we get from the first term:

$$r^2R''(r)+rR'(r)=0 \tag{3}$$

This has the solution

$$R(r)=A\ln r+B \tag{4}$$

as can be verified by direct substitution. From the second term

$$\Phi(\phi)=C\phi+D \tag{5}$$

However, we need to remember that ϕ is an angle, and is restricted to $[0,2\pi]$, so the $C\phi$ term doesn't behave properly, in that it's not periodic. So we need to take $C=0$, giving $\Phi=constant$ as the only solution in this case.

If we now consider the more general case, then the angular equation is, taking the constant as $-k^2$ as in previous applications of separation of variables:

$$\Phi''(\phi)=-k^2\Phi \tag{6}$$

which has the general solution

$$\Phi(\phi) = C\sin k\phi + D\cos k\phi \tag{7}$$

The radial equation now becomes

$$r^2 R''(r) + rR'(r) - k^2 R(r) = 0 \tag{8}$$

This has the general solution

$$R = \sum_{n=1}^{\infty} a_n r^n \tag{9}$$

Substituting into the ODE, we get

$$\sum_{n=1}^{\infty} \left[a_n n(n-1) + a_n n - a_n k^2\right] r^n = 0 \tag{10}$$

From the uniqueness of power series, the coefficient of each power of r must be zero, from which we get

$$a_n(n^2 - k^2) = 0 \tag{11}$$

Thus either $n = \pm k$ or $a_n = 0$. This means that k must be an integer, and for a given choice of $k = n$, the solution is

$$R_n(r) = a_n r^n \tag{12}$$

The general solution is the linear combination of all the particular solutions, so we get (redefining the constants):

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty} r^n (A_n \sin n\phi + B_n \cos n\phi) + \sum_{n=-\infty}^{-1} r^n (C_n \sin n\phi + D_n \cos n\phi) \tag{13}$$

For example, in the case of an infinite wire, V is independent of ϕ so we get

$$V(r) = A\ln r + B \tag{14}$$

As we've seen earlier (Example 2 in this post), to nail this down any more requires that we specify a reference point (or surface) where $V = 0$, and for this case it doesn't really matter as long as we avoid $r = 0$ and $r = \infty$.

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