LAPLACE'S EQUATION IN SPHERICAL COORDINATES

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In spherical coordinates, Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0 \tag{1}$$

We can cancel off the factor of r^2. If we further assume that V is independent of ϕ, as is true for many problems, the last term disappears (we'll hopefully get around to considering the general case eventually), and we are left with

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0 \tag{2}$$

We can try the usual technique of separation of variables, and assume that

$$V(r,\theta)=R(r)\Theta(\theta) \tag{3}$$

Plugging this into the equation and dividing through by V, we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0 \tag{4}$$

The argument that we used in the rectangular coordinate case applies here too: since each term on the left depends on only one of the two independent variables (r and θ), each term must be a constant. After solving the equation, in hindsight it turns out that the best way to write this constant is:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1) \tag{5}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1) \tag{6}$$

The general solution of the radial equation is

$$R(r) = Ar^l + \frac{B}{r^{l+1}} \tag{7}$$

as may be verified by direct substitution. Note that here, the constant l can be any real number; it's not restricted to being an integer.

The angular equation is more complex, but we've already considered it. As we saw there, the solutions are the Legendre polynomials. Moreover, as we worked through the solution of this equation to get the Legendre polynomials, we found that the only physically acceptable solutions (that is, ones that don't become infinite at some point) are those for which l is a positive integer. So we get

$$\Theta(\theta) = P_l(\cos\theta) \tag{8}$$

where P_l is the Legendre polynomial corresponding to a particular integer l. It turns out that P_l is a polynomial of degree l in its argument.

The polynomials can be generated by using the Rodrigues formula, as we saw earlier. The Rodrigues formula also allowed us to derive an orthogonality property of the Legendre polynomials, that is:

$$\int_0^\pi P_l(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \begin{cases} 0 & \text{if } l \neq m \\ \frac{2}{2l+1} & \text{if } l = m \end{cases} \tag{9}$$

Thus, for a particular value of l, the solution to Laplace's equation is

$$V_l(r,\theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \tag{10}$$

where A_l and B_l are constants to be determined by the boundary conditions of the particular problem.

As before, since Laplace's equation is linear, we can form a general solution by summing up the particular solutions for all the values of l.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \tag{11}$$

Example 1. The standard problem for illustrating how this general formula can be used is that of a hollow sphere of radius R, on which a potential $V_R(\theta)$ that depends only on θ is specified. The problem is to find the potential inside and outside the sphere, assuming no other charge is present.

For the inside problem, we must have $B_l = 0$ for all l, to prevent an infinity at the origin. On the boundary (that is, the sphere), we know the potential to be $V_R(\theta)$, so we have a boundary condition:

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_R(\theta) \tag{12}$$

We can now use the orthogonality condition 9 to find an expression for the coefficients A_l. If we multiply this equation through by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π we get

$$\sum_{l=0}^{\infty} \int_0^{\pi} A_l R^l P_l(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = \int_0^{\pi} V_R(\theta) P_m(\cos\theta)\sin\theta d\theta \tag{13}$$

$$A_m R^m \frac{2}{2m+1} = \int_0^{\pi} V_R(\theta) P_m(\cos\theta)\sin\theta d\theta \tag{14}$$

$$A_m = \frac{2m+1}{2R^m} \int_0^{\pi} V_R(\theta) P_m(\cos\theta)\sin\theta d\theta \tag{15}$$

In the second line, we are able to eliminate all terms from the sum where $l \neq m$ by using the orthogonality condition. To proceed any further, we would need to know the function $V_R(\theta)$.

For the outside problem, we can set all $A_l = 0$ to prevent V from becoming infinite for large r. This time the general solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{16}$$

At the boundary, we get

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_R(\theta) \tag{17}$$

We can use exactly the same procedure to find the coefficients B_l. Multiply through by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π and we get

$$\frac{B_m}{R^{m+1}} \frac{2}{2m+1} = \int_0^{\pi} V_R(\theta) P_m(\cos\theta)\sin\theta d\theta \tag{18}$$

$$B_m = \frac{(2m+1)R^{m+1}}{2} \int_0^{\pi} V_R(\theta) P_m(\cos\theta)\sin\theta d\theta \tag{19}$$

where again we need an explicit form for $V_R(\theta)$ to proceed.

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