

LIÉNARD'S GENERALIZATION OF THE LARMOR FORMULA FOR AN ACCELERATING CHARGE

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For an accelerating charge q that is instantaneously at rest, the power radiated is given by the Larmor formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \quad (1)$$

where the acceleration a is a function of time. Griffiths shows in his section 11.2.1 that the Larmor formula is the integral of the Poynting vector over a large sphere. Another way of looking at it is that the charge radiates an amount of power dP into an element of solid angle $d\Omega = \sin\theta d\theta d\phi$ and the formula for this is obtained from the Poynting vector by multiplying by r^2 to make the result independent of distance from the charge.

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \sin^2\theta \quad (2)$$

We can see from this formula that the angle at which the maximum power is radiated is $\theta_{max} = \frac{\pi}{2}$.

To generalize these formulas to the case where $v \neq 0$ requires a bit of a slog through the mathematics, but the results are quoted by Griffiths as his equations 11.72 and 11.73:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c^2} \frac{|\hat{\mathbf{t}} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{\mathbf{t}} \cdot \mathbf{u})^5} \quad (3)$$

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left[a^2 - \frac{|\mathbf{v} \times \mathbf{a}|^2}{c^2} \right] \quad (4)$$

where $\hat{\mathbf{t}}$ is a unit vector pointing from the charge at the retarded time to the observation point on the enclosing sphere and $\mathbf{u} = c\hat{\mathbf{t}} - \mathbf{v}$. These formulas are Liénard's generalization of the Larmor formula.

For a charge with $\mathbf{v} \parallel \mathbf{a}$ (at the instant of retarded time), Griffiths shows in his Example 11.3 that

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5} \quad (5)$$

where $\beta \equiv v/c$. This formula reduces to 2 when $\beta = 0$.

We can find the angle θ_{max} for the case $\beta \neq 0$ by differentiating 5 and setting the result to 0.

$$\frac{d}{d\theta} \frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \left[2 \frac{\sin(\theta) \cos(\theta)}{(1 - \beta \cos(\theta))^5} - 5 \frac{(\sin(\theta))^3 \beta}{(1 - \beta \cos(\theta))^6} \right] = 0 \quad (6)$$

The solution $\theta = 0$ gives the angle of *minimum* power, so if we take $\theta \neq 0$ we can cancel off $\sin \theta$ and then multiply through by $(1 - \beta \cos(\theta))^6$ to get

$$2(1 - \beta \cos(\theta)) \cos(\theta) - 5(1 - (\cos(\theta))^2) \beta = 0 \quad (7)$$

which has the solution

$$\theta_{max} = \arccos \left[\frac{\sqrt{15\beta^2 + 1} - 1}{3\beta} \right] \quad (8)$$

In the ultrarelativistic case, we can write

$$\beta = 1 - x \quad (9)$$

where $x \ll 1$ and expand in a Taylor series (using Maple to do the heavy lifting):

$$\frac{\sqrt{15\beta^2 + 1} - 1}{3\beta} = \frac{\sqrt{15(1-x)^2 + 1} - 1}{3(1-x)} \quad (10)$$

$$= 1 - \frac{x}{4} - \frac{27}{128}x^2 + \mathcal{O}(x^3) \quad (11)$$

Since x is small, the series on the right is very close to 1, which means that the argument of the arccos is close to 1, so θ_{max} is close to 0, so we can approximate

$$\cos \theta_{max} \approx 1 - \frac{\theta_{max}^2}{2} \quad (12)$$

$$\approx 1 - \frac{x}{4} \quad (13)$$

$$\theta_{max} \approx \sqrt{\frac{x}{2}} \quad (14)$$

$$= \sqrt{\frac{1-\beta}{2}} \quad (15)$$

To compare the power output at the maximum angles in the two cases, we have from 2

$$\frac{dP}{d\Omega_{rest}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \quad (16)$$

and from 5

$$\frac{dP}{d\Omega_{rel}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2 \theta_{max}}{(1 - \beta \cos \theta_{max})^5} \quad (17)$$

so the ratio is

$$\frac{dP/d\Omega_{rel}}{dP/d\Omega_{rest}} = \frac{\sin^2 \theta_{max}}{(1 - \beta \cos \theta_{max})^5} \quad (18)$$

For $\theta_{max} \approx \sqrt{\frac{1-\beta}{2}} \ll 1$ we can approximate

$$\sin^2 \theta_{max} \approx \frac{1-\beta}{2} \quad (19)$$

$$= \frac{x}{2} \quad (20)$$

$$1 - \beta \cos \theta_{max} \approx 1 - \beta \left(1 - \frac{1-\beta}{4}\right) \quad (21)$$

$$\approx 1 - (1-x) \left(1 - \frac{x}{4}\right) \quad (22)$$

$$\approx \frac{5x}{4} \quad (23)$$

Therefore

$$\frac{dP/d\Omega_{rel}}{dP/d\Omega_{rest}} \approx \frac{4^5 x}{2(5x)^5} \quad (24)$$

$$= \frac{512}{3125(1-\beta)^4} \quad (25)$$

To express this in terms of the relativistic factor $\gamma = 1/\sqrt{1-\beta^2}$ we can approximate γ for $\beta = 1 - x$:

$$\gamma = \frac{1}{\sqrt{1-(1-x)^2}} \quad (26)$$

$$= \frac{1}{\sqrt{2x-x^2}} \quad (27)$$

$$\approx \frac{1}{\sqrt{2x}} \quad (28)$$

$$= \frac{1}{\sqrt{2(1-\beta)}} \quad (29)$$

Therefore

$$\frac{1}{(1-\beta)^4} \approx 16\gamma^8 \quad (30)$$

$$\frac{dP/d\Omega_{rel}}{dP/d\Omega_{rest}} \approx 2.62\gamma^8 \quad (31)$$

Since γ gets very large for $\beta \approx 1$, the power generated by a relativistic charge is enormously greater than that of a charge at rest.

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