LIÉNARD-WIECHERT POTENTIAL FOR A CHARGE MOVING ON A HYPERBOLIC TRAJECTORY

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We can now work out the Liénard-Wiechert potentials for a point charge moving on a hyperbolic trajectory. We're trying to find

$$V(\mathbf{r},t)=\frac{qc}{4\pi\epsilon_0}\frac{1}{(c\left|\mathbf{r}-\mathbf{w}(t_r)\right|-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v})} \tag{1}$$

The motion is in one dimension, given by

$$\mathbf{w}(t)=\sqrt{b^2+c^2t^2}\hat{\mathbf{x}} \tag{2}$$

It's worth noting here that $w\ge b>0$ for all times, and the velocity is

$$\mathbf{v}(t)=\frac{d\mathbf{w}}{dt}=\frac{c^2t}{\sqrt{b^2+c^2t^2}}\hat{\mathbf{x}} \tag{3}$$

which is negative for $t<0$ (when the particle is moving in from the right) and positive for $t>0$ (when the particle is moving back out again).

We'll consider only observation points $\mathbf{r}$ that lie on the x axis to the right of the particle's location, so the separation is

$$d(t)=r-w(t)=x-\sqrt{b^2+c^2t^2}>0 \tag{4}$$

Using this notation, the potential is

$$V(\mathbf{r},t)=\frac{qc}{4\pi\epsilon_0}\frac{1}{(cd(t_r)-\mathbf{d}(t_r)\cdot\mathbf{v})} \tag{5}$$

The retarded time is given by

$$d(t_r)=c(t-t_r) \tag{6}$$

$$x-\sqrt{b^2+c^2t_r^2}=c(t-t_r) \tag{7}$$

We can solve this by isolating the square root on one side and then squaring both sides, to get

$$t_r = \frac{x^2 - 2xct + c^2t^2 - b^2}{2c(ct - x)} \tag{8}$$

Note that for $t = 0$, as $x \to \infty$, $t_r \to -\infty$ which makes sense, since the further out on the x axis we place the observer, the further back in time we need to go to get a signal from the particle. We can substitute this back into 2 to get

$$w(t_r) = \frac{1}{2}\sqrt{\frac{\left(x^2 - 2xct + c^2t^2 + b^2\right)^2}{(ct - x)^2}} \tag{9}$$

Although the operand of the square root is a perfect square, we need to be careful when taking the square root to ensure we get the correct sign. Since $w > 0$ we can look at $t = 0$ as before, at which point

$$w(t_r) = \frac{1}{2}\sqrt{\frac{\left(x^2 + b^2\right)^2}{(-x)^2}} > 0 \tag{10}$$

so we need to take the negative root of the operand to get $w > 0$. That is

$$w(t_r) = \frac{x^2 - 2xct + c^2t^2 + b^2}{2(x - tc)} \tag{11}$$

We can now calculate

$$d(t_r) = x - w(t_r) \tag{12}$$

$$= \frac{x^2 - c^2t^2 - b^2}{2(x - ct)} \tag{13}$$

after simplifying.

Now we need to find $\mathbf{v}(t_r)$. Substituting 8 into 3 (I used Maple for the following calculations):

$$\mathbf{v}(t_r) = -\frac{c\left(b^2 - x^2 + 2xct - c^2t^2\right)}{2(ct - x)\sqrt{b^2 + \frac{\left(b^2 - x^2 + 2xct - c^2t^2\right)^2}{4(-x + ct)^2}}} \tag{14}$$

Simplifying the operand of the square root, we get

$$\mathbf{v}(t_r) = -\frac{c\left(b^2 - x^2 + 2xct - c^2t^2\right)}{2(ct - x)\sqrt{\frac{\left(x^2 - 2xct + c^2t^2 + b^2\right)^2}{4(ct - x)^2}}} \tag{15}$$

Again, we need to take the correct sign when taking the square root. For $t = 0$ we get

$$\mathbf{v}(t_r) = c\frac{b^2 - x^2}{2x\sqrt{\frac{\left(x^2+b^2\right)^2}{4(-x)^2}}} \tag{16}$$

For large x, the signal comes from the particle in the distant past when it was moving to the left, so we should have v negative in this case. This again requires taking the negative root, so we get

$$\mathbf{v}(t_r) = \frac{c\left(b^2 - x^2 + 2\,xct - c^2t^2\right)}{x^2 - 2xct + c^2t^2 + b^2} \tag{17}$$

Putting it all together, we get

$$cd(t_r) - \mathbf{d}(t_r)\cdot\mathbf{v} = \frac{1}{2}\,\frac{\left(-x^2 + c^2t^2 + b^2\right)c}{-x + ct} - \frac{1}{2}\,\frac{\left(-x^2 + c^2t^2 + b^2\right)c\left(b^2 - (-x + ct)^2\right)}{(-x + ct)\left(b^2 + (-x + ct)^2\right)} \tag{18}$$

$$= \frac{(-x + ct)\left(-x^2 + c^2t^2 + b^2\right)c}{x^2 - 2\,xct + c^2t^2 + b^2} \tag{19}$$

Therefore the potential is

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\,\frac{q\left(x^2 - 2\,xct + c^2t^2 + b^2\right)}{(-x + ct)\left(-x^2 + c^2t^2 + b^2\right)} \tag{20}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{q\left((x - ct)^2 + b^2\right)}{(x - ct)\left(x^2 - c^2t^2 - b^2\right)} \tag{21}$$