LIÉNARD-WIECHERT POTENTIALS FOR A CHARGE MOVING WITH CONSTANT VELOCITY

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Griffiths shows in his example 10.3 that the Liénard-Wiechert potentials for a point charge q moving at constant velocity $\mathbf{v}$ that passes through the origin at time $t=0$ are

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}} \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}} \tag{2}$$

These potentials can be expressed in a simpler form by defining the vector

$$\mathbf{R} \equiv \mathbf{r}-\mathbf{v}t \tag{3}$$

We can eliminate $\mathbf{r}$ from 1 as follows.

$$\mathbf{R}\cdot\mathbf{v} = \mathbf{r}\cdot\mathbf{v}-v^2t \tag{4}$$

$$\mathbf{r}\cdot\mathbf{v} = \mathbf{R}\cdot\mathbf{v}+v^2t \tag{5}$$

$$R^2 = r^2+v^2t^2-2\mathbf{r}\cdot\mathbf{v}t \tag{6}$$

$$r^2 = R^2-v^2t^2+2\mathbf{r}\cdot\mathbf{v}t \tag{7}$$

$$= R^2+v^2t^2+2\mathbf{R}\cdot\mathbf{v}t \tag{8}$$

$$\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2 = \left(c^2t-\mathbf{R}\cdot\mathbf{v}-v^2t\right)^2 \tag{9}$$

$$\left(c^2-v^2\right)\left(r^2-c^2t^2\right) = \left(c^2-v^2\right)\left(R^2+v^2t^2+2\mathbf{R}\cdot\mathbf{v}t-c^2t^2\right) \tag{10}$$

Adding the last two RHSs together and cancelling terms, we get

$$\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right) = (\mathbf{R}\cdot\mathbf{v})^2+\left(c^2-v^2\right)R^2 \tag{11}$$

If θ is the angle between $\mathbf{R}$ and $\mathbf{v}$, then this becomes

$$(\mathbf{R}\cdot\mathbf{v})^2+\left(c^2-v^2\right)R^2=R^2\left(c^2-v^2\left(1-\cos^2\theta\right)\right) \tag{12}$$

$$=R^2c^2\left(1-\frac{v^2}{c^2}\sin^2\theta\right) \tag{13}$$

Inserting this back into 1 we get

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)}} \tag{14}$$

Note that R and θ are both functions of time since they vary as the charge moves. For non-relativistic speeds, $v\ll c$ and the formula reduces to the Coulomb potential from electrostatics:

$$V=\frac{q}{4\pi\epsilon_0 R} \tag{15}$$