LINEAR CHARGE DISTRIBUTIONS

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When faced with a continuous distribution of charge, we can work out the electric field as a function of position by using integration instead of summation. In general, we have

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

Here $\mathbf{r}'$ is the position of volume element $d^3\mathbf{r}'$ and $\rho(\mathbf{r}')$ is the charge density at that point. There are three types of problems that occur commonly with continuous charge distributions: linear, surface and volume charges. We'll do a few examples using linear charges here to see how this works in practice. In many problems in electrostatics, it's advisable to make use of any symmetries that the configuration has.

Example 1. Suppose we have a line segment extending along the x axis from $-L$ to L. This line segment contains a constant linear charge density λ (measured in Coulombs/metre). What is the electric field at a point z on the z axis?

We can split the problem in two by solving for the x and z components of $\mathbf{E}$ separately. Because the z axis divides the linear charge precisely in two, we can use symmetry to conclude that there is no net x component in the field. To work out the z component, we note that the contributions from $+x$ and $-x$ are equal.

In the formula above, $\mathbf{r}-\mathbf{r}'$ is the vector from a point on the linear charge to the point z so we get

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{x^2+z^2} \tag{2}$$

The unit vector $\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$ has a z component of $z/\sqrt{x^2+z^2}$ and the charge density is $\rho=\lambda$ so we get for the magnitude of $\mathbf{E}$ in the z direction:

$$E_z = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{2\lambda z}{(x^2+z^2)^{3/2}}dx \tag{3}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{L^2+z^2}} \tag{4}$$

where you can either work out the integral by hand or look it up or use software like Maple.

Note that for $z \gg L$, we get

$$E_z \to \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z^2} \tag{5}$$

which is equivalent to the field of a point charge $q = 2\lambda L$ at a distance z.

For $L \gg z$ we get

$$E_z \to \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z} \tag{6}$$

which is the formula for the field due to an infinitely long line of charge.

Example 2. A slight variant on this problem is to remove one half of the line segment, so the linear charge now extends from $x = 0$ to $x = L$, with the test point still on the z axis. The z component of the field will now be half that calculated above:

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z\sqrt{L^2+z^2}} \tag{7}$$

However, since the problem is no longer symmetric about the origin, E_x is no longer zero. The x component of $\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$ is $-x/\sqrt{x^2+z^2}$ (the negative sign arises because the vector points *from* $\mathbf{r}'$ *to* $\mathbf{r}$, and since all $\mathbf{r}'$ locations are on the $+x$ axis, and $\mathbf{r}$ is on the z axis, the x component of $\mathbf{r}-\mathbf{r}'$ is always negative) so we get

$$E_x = -\frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda x}{(x^2+z^2)^{3/2}}dx \tag{8}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\frac{z-\sqrt{L^2+z^2}}{z\sqrt{L^2+z^2}} \tag{9}$$

For $z \gg L$:

$$E_z \to \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2} \tag{10}$$

$$E_x \to 0 \tag{11}$$

Example 3. We now have a square loop (like a wire bent into a square) lying in the xy plane with sides parallel to the axes and centred at the origin, and we want to find the field at some point z on the z axis. By symmetry, the field will be entirely along the z direction, and the contributions from all four sides will be equal. If we consider the edge where $y = a/2$, then the distance from a point on this edge to z is $\sqrt{x^2 + \frac{a^2}{4} + z^2}$. Using the same reasoning as in example 1, the z component of the unit vector connecting this point with z is $z/\sqrt{x^2 + \frac{a^2}{4} + z^2}$ so we get

$$E_z = \frac{1}{4\pi\epsilon_0}\int_{-a/2}^{a/2}\frac{4\lambda z}{\left(x^2 + \frac{a^2}{4} + z^2\right)^{3/2}}dx \tag{12}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{8\lambda a z}{\sqrt{2a^2 + 4z^2}\left(z^2 + \frac{a^2}{4}\right)} \tag{13}$$

As a check, when $z \gg a$, we get

$$E_z \rightarrow \frac{1}{4\pi\epsilon_0}\frac{4\lambda a}{z^2} \tag{14}$$

which is the equivalent of the field due to a point charge $q = 4\lambda a$.

Example 4. Now consider a circular loop of radius r in the xy plane, centred at the origin. By symmetry, the field is entirely in the z direction. A line segment on the circle has length $r d\theta$, where θ is the angle in the xy plane. The contribution from all line segments is the same. The z component of the unit vector is $z/\sqrt{z^2 + r^2}$ so the field is

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda r z}{(z^2 + r^2)^{3/2}}\int_0^{2\pi} d\theta \tag{15}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2\pi r \lambda z}{(z^2 + r^2)^{3/2}} \tag{16}$$

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