LORENZ GAUGE IS ALWAYS POSSIBLE

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The Lorenz gauge is defined by setting

$$\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t} \tag{1}$$

but is it always possible to do this? We can show that it is using a similar technique to that for the Coulomb gauge. We want a function λ which we can use to transform some arbitrary potentials $\mathbf{A}'$ and V so that that $\mathbf{A}$ satisfies 1 as follows:

$$\mathbf{A}=\mathbf{A}'+\nabla\lambda \tag{2}$$

$$V=V'-\frac{\partial\lambda}{\partial t} \tag{3}$$

Taking the divergence of both sides, we get

$$\nabla\cdot\mathbf{A}=\nabla\cdot\mathbf{A}'+\nabla^2\lambda \tag{4}$$

$$=-\mu_0\epsilon_0\frac{\partial V}{\partial t} \tag{5}$$

$$=-\mu_0\epsilon_0\frac{\partial V'}{\partial t}+\mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} \tag{6}$$

Combining the first and last equations we get

$$\nabla^2\lambda-\mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2}=-\mu_0\epsilon_0\frac{\partial V'}{\partial t}-\nabla\cdot\mathbf{A}' \tag{7}$$

$$\square^2\lambda=-\mu_0\epsilon_0\frac{\partial V'}{\partial t}-\nabla\cdot\mathbf{A}' \tag{8}$$

That is, λ is the solution of the wave equation with a driving term, which we can, in principle, always solve (although it may not be easy!). Therefore we can always find the function λ to convert an arbitrary pair of potentials $\mathbf{A}'$ and V' to the Lorenz gauge.

In general (not necessarily in the Lorenz gauge), we could always set $V=0$ by choosing

$$\frac{\partial\lambda}{\partial t} = V' \tag{9}$$

$$\lambda = \int_0^t V'\left(\mathbf{r}, t'\right) dt' \tag{10}$$

We can't always choose $\mathbf{A} = 0$ however, since $\mathbf{B} = \nabla \times \mathbf{A}$ and if the magnetic field is non-zero, then $\mathbf{A}$ can't be zero everywhere.