MAGNETIC DIPOLE

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Post date: 15 Mar 2021.

In much the same way as the electric potential, we can write the magnetic potential as a multipole expansion. Because the magnetic potential is a vector, the expansion is a series of vector integrals rather than volume integrals.

The most general form of the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \tag{1}$$

For a steady line current I in a closed loop, this becomes a line integral around the loop:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{l}' \tag{2}$$

We can expand the integrand in terms of Legendre polynomials in the same way as for the electric potential:

$$\begin{aligned}
\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} &(3)\\
&= \frac{1}{r}\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta'}} &(4)\\
&= \frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta')\left(\frac{r'}{r}\right)^n &(5)
\end{aligned}$$

The potential becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint P_n(\cos\theta') r'^n d\mathbf{l}' \tag{6}$$

The first three terms in this sum are called the monopole, dipole and quadrupole terms. The monopole term is

$$\mathbf{A}_0 = \frac{\mu_0 I}{4\pi r}\oint d\mathbf{l}' = 0 \tag{7}$$

since integration of the vector line element around any closed loop is zero. For the dipole:

$$\mathbf{A}_1 = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta' d\mathbf{l}' \tag{8}$$

Since θ' is the angle between $\mathbf{r}'$ and $\mathbf{r}$, we can write this as

$$\mathbf{A}_1 = \frac{\mu_0 I}{4\pi r^2}\oint \mathbf{r}'\cdot\hat{\mathbf{r}} d\mathbf{l}' \tag{9}$$

Since $\hat{\mathbf{r}}$ is a constant as far as the integral is concerned, we can write the integral in terms of the vector area $\mathbf{a}$ enclosed by the loop.

$$\mathbf{A}_1 = \frac{\mu_0 I}{4\pi r^2}\mathbf{a}\times\hat{\mathbf{r}} \tag{10}$$

The quantity

$$\mathbf{m} \equiv I\mathbf{a} \tag{11}$$

is defined as the *magnetic dipole moment* of the current loop, so the dipole potential is

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi r^2}\mathbf{m}\times\hat{\mathbf{r}} \tag{12}$$

The magnetic field due to a dipole is

$$\mathbf{B}_1 = \nabla\times\mathbf{A}_1 \tag{13}$$

$$= \frac{\mu_0}{4\pi}\nabla\times\left(\mathbf{m}\times\frac{\hat{\mathbf{r}}}{r^2}\right) \tag{14}$$

We can use a vector identity to expand the curl, and use the fact that $\mathbf{m}$ is a constant for a given current loop, so its derivatives are all zero. We get

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi}\left[\mathbf{m}\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) - (\mathbf{m}\cdot\nabla)\frac{\hat{\mathbf{r}}}{r^2}\right] \tag{15}$$

We can now use

$$\hat{\mathbf{r}} = \frac{1}{r}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \tag{16}$$

$$r = \sqrt{x^2+y^2+z^2} \tag{17}$$

Therefore

$$\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2} = \partial_x\frac{x}{r^3}+\partial_y\frac{y}{r^3}+\partial_z\frac{z}{r^3} \tag{18}$$

$$= \left(-\frac{3}{2}\frac{2x^2}{r^5}+\frac{1}{r^3}\right)+\left(-\frac{3}{2}\frac{2y^2}{r^5}+\frac{1}{r^3}\right)+\left(-\frac{3}{2}\frac{2z^2}{r^5}+\frac{1}{r^3}\right) \tag{19}$$

$$= 0 \tag{20}$$

$$(\mathbf{m}\cdot\nabla)\frac{\hat{\mathbf{r}}}{r^2} = m_x\left[\left(-\frac{3}{2}\frac{2x^2}{r^5}+\frac{1}{r^3}\right)\hat{\mathbf{x}}-\frac{3}{2}\frac{2xy}{r^5}\hat{\mathbf{y}}-\frac{3}{2}\frac{2xz}{r^5}\hat{\mathbf{z}}\right]+m_y[\ldots]+m_z[\ldots] \tag{21}$$

$$= -3\frac{xm_x}{r^5}\mathbf{r}+\frac{m_x}{r^3}\hat{\mathbf{x}}-3\frac{ym_y}{r^5}\mathbf{r}+\frac{m_y}{r^3}\hat{\mathbf{y}}-3\frac{zm_z}{r^5}\mathbf{r}+\frac{m_z}{r^3}\hat{\mathbf{z}} \tag{22}$$

$$= \frac{1}{r^3}\left[-3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}+\mathbf{m}\right] \tag{23}$$

Putting this all together, we get

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right] \tag{24}$$

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