MAGNETIC FIELD - UNIQUENESS CONDITIONS

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We've seen that solutions to Laplace's and Poisson's equations are unique, so it's natural to ask if a similar condition exists in the case of the magnetic field and its vector potential. That is, suppose we specify the field **B** or the potential **A** on the boundary of some volume, and also specify the current density **J** within that volume. Does this determine the field uniquely within the volume?
First, we need an identity which can be derived from the divergence theorem and a vector calculus identity. For some vector fields **A** and **B**, we have the identity

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\times(\nabla\times\boldsymbol{A})-\boldsymbol{A}\times(\nabla\times\boldsymbol{B}) \tag{1}$$

Using this with $\boldsymbol{A}=\boldsymbol{U}$ and $\boldsymbol{B}=\nabla\times\boldsymbol{V}$, we get

$$\int_A\mathbf{U}\times(\nabla\times\mathbf{V})\cdot d\mathbf{a}=\int_V\nabla\cdot[\mathbf{U}\times(\nabla\times\mathbf{V})]\,d^3\mathbf{r} \tag{2}$$

$$=\int_V[(\nabla\times\mathbf{U})\cdot(\nabla\times\mathbf{V})-\mathbf{U}\cdot(\nabla\times(\nabla\times\mathbf{V}))]\,d^3\mathbf{r} \tag{3}$$

Following the same logic as in the electrostatic case, we suppose that there are two different solutions $\mathbf{B}_1$ and $\mathbf{B}_2$ (with corresponding potentials $\mathbf{A}_1$ and $\mathbf{A}_2$) and consider the difference $\mathbf{B}_3\equiv\mathbf{B}_2-\mathbf{B}_1$. Because the curl operator is linear, we have

$$\nabla\times\mathbf{A}_3=\nabla\times(\mathbf{A}_2-\mathbf{A}_1) \tag{4}$$

$$=\mathbf{B}_2-\mathbf{B}_1 \tag{5}$$

$$=\mathbf{B}_3 \tag{6}$$

Setting $\mathbf{U}=\mathbf{V}=\mathbf{A}_3$ in the identity 3, we get on the LHS

$$\int_A\mathbf{U}\times(\nabla\times\mathbf{V})\cdot d\mathbf{a}=\int_A(\mathbf{A}_3\times\mathbf{B}_3)\cdot d\mathbf{a} \tag{7}$$

Note that if we specify either **A** or **B** on the boundary surface, then either $\mathbf{A}_3 = 0$ or $\mathbf{B}_3 = 0$ since there can be either a unique potential or a unique field on the boundary. Thus this surface integral is always zero.
It's important to note here that specifying **A** on the boundary only does *not* specify **B** on the boundary as well, since we need to know **A** in three dimensions in order to calculate its curl.
On the RHS of 3, we get, from Ampère's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$:

$$0 = \int_V [\mathbf{B}_3 \cdot \mathbf{B}_3 - \mu_0 \mathbf{A}_3 \cdot \mathbf{J}_3]\, d^3\mathbf{r} \tag{8}$$

Since we've specified the current inside the volume $\mathbf{J}_3 = \mathbf{J}_2 - \mathbf{J}_1 = 0$ so we get

$$\int_V B_3^2 d^3\mathbf{r} = 0 \tag{9}$$

Since the integral of a positive definite quantity over the volume is zero, it must be zero everywhere, so $\mathbf{B}_3 = 0$ and $\mathbf{B}_2 = \mathbf{B}_1$. So, yes, the field is determined uniquely within the volume.