MAGNETIC FIELD OF ROTATING SPHERE OF CHARGE

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Since any moving charge generates a magnetic field, one way of producing a novel current is to take a uniform sphere of charge and set it spinning on its axis. To work out the field produced by such a sphere, we can start with the field generated by a spinning spherical shell of charge. The derivation of this field is surprisingly tricky, and is given by Griffiths in his EM book as example 5.11. The results for the vector potential are

$$\mathbf{A}(r,\theta,\phi)=\begin{cases}\frac{1}{3}\mu_0 R\omega\sigma r\sin\theta\hat{\boldsymbol{\phi}} & r\le R\\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}} & r\ge R\end{cases}\tag{1}$$

Here, R is the radius of the shell, σ is the surface charge density and ω is the angular velocity, where the sphere's axis is taken to be the z axis. The calculation is done using spherical coordinates.

From this we can calculate the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ using the standard formula for the curl in spherical coordinates:

$$\mathbf{B}=\begin{cases}\frac{2\mu_0 R\omega\sigma}{3}\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}\right) & r\le R\\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{1}{r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right) & r\ge R\end{cases}\tag{2}$$

Notice that although $\mathbf{A}$ is continuous at $r=R$, $\mathbf{B}$ is not. It's also worth noting that since $\hat{\mathbf{z}}=\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}$, the field inside the shell points along the rotation axis and is uniform.

To find the field due to a solid spinning sphere of charge with charge density ρ, we can integrate a series of spherical shells. In doing this, we need to be very careful in interpreting the various symbols for the radius.

For a thin shell of thickness dr, the charge per unit area on this shell is ρdr, so this will replace σ in the equations above. In these equations, R is the radius of the shell, and r is the observation radius. We need the overall radius of the solid sphere, which we'll define as R_0. Thus the integration variable will be R, since it is the radius of the shells that we need to vary.

Then for a value of r inside the sphere, we will get a contribution to the field from those shells with a radius $R<r$ by using the second equation in 2, and for $R>r$ by using the first equation. Note that we do *not* integrate over either r or θ, since these two coordinates define the observation point. We get for the field due to shells interior to the observation radius:

$$\mathbf{B}_{\text{in}} = \frac{\mu_0\omega\rho}{3}\frac{1}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)\int_0^r R^4 dR \tag{3}$$

$$= \frac{\mu_0\omega\rho}{3}\frac{r^2}{5}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \tag{4}$$

From shells outside the observation radius:

$$\mathbf{B}_{\text{out}} = \frac{2\mu_0\omega\rho}{3}\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right)\int_r^{R_0} RdR \tag{5}$$

$$= \frac{\mu_0\omega\rho}{3}\left(R_0^2 - r^2\right)\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right) \tag{6}$$

The total field is the sum so

$$\mathbf{B} = \frac{\mu_0\omega\rho}{3}\left[\left(R_0^2 - \frac{3}{5}r^2\right)\cos\theta\hat{\mathbf{r}} + \left(\frac{6}{5}r^2 - R_0^2\right)\sin\theta\hat{\boldsymbol{\theta}}\right] \tag{7}$$

If we know the total charge Q in the sphere, then $\rho = \frac{3Q}{4\pi R_0^3}$ and

$$\mathbf{B} = \frac{\mu_0\omega Q}{4\pi R_0}\left[\left(1 - \frac{3}{5}\frac{r^2}{R_0^2}\right)\cos\theta\hat{\mathbf{r}} + \left(\frac{6}{5}\frac{r^2}{R_0^2} - 1\right)\sin\theta\hat{\boldsymbol{\theta}}\right] \tag{8}$$

$$= \frac{\mu_0\omega Q}{4\pi R_0}\left[\hat{\mathbf{z}} - \frac{3}{5}\frac{r^2}{R_0^2}\cos\theta\hat{\mathbf{r}} + \frac{6}{5}\frac{r^2}{R_0^2}\sin\theta\hat{\boldsymbol{\theta}}\right] \tag{9}$$

Although 9 mixes spherical and rectangular coordinates, it shows that there is a uniform field in the z direction with a varying field superimposed on top of it.