MAGNETIC VECTOR POTENTIAL

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The Biot-Savart law gives the magnetic field **B** in terms of the currents in a volume:

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_V\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' \tag{1}$$

By straightforward calculation, we can show that

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}=-\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{2}$$

where the gradient is with respect to $\boldsymbol{r}$, not $\boldsymbol{r}'$.

Plugging this into the integral, we get

$$\mathbf{B}(\mathbf{r})=-\frac{\mu_0}{4\pi}\int_V\mathbf{J}\left(\mathbf{r}'\right)\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{3}$$

The gradient operator operates on the unprimed coordinates only, so we can rewrite the integrand as

$$-\mathbf{J}\left(\mathbf{r}'\right)\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\nabla\times\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4}$$

To see that this works, consider the general equation

$$-\mathbf{J}\times\nabla\Phi=\nabla\times(\mathbf{J}\Phi) \tag{5}$$

where on both sides, the ∇ operator operates only on Φ and not on **J**. Looking at each side separately and isolating the x component, we get

$$[-\mathbf{J}\times\nabla\Phi]_x=-J_y\partial_z\Phi+J_z\partial_y\Phi \tag{6}$$

$$[\nabla\times(\mathbf{J}\Phi)]_x=\partial_y(J_z\Phi)-\partial_z(J_y\Phi) \tag{7}$$

$$=J_z\partial_y\Phi-J_y\partial_z\Phi \tag{8}$$

where in the last line we can pull the components of **J** outside the derivatives since it depends only on the primed coordinates. The same derivation works for the y and z components as well. Therefore we can rewrite the Biot-Savart law as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{9}$$

The vector quantity $\mathbf{A}(\mathbf{r})$ is defined by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{10}$$

and is known as the *magnetic vector potential*. This is a magnetic analog of the electrostatic condition $\mathbf{E} = -\nabla\Phi$, as we can write

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{11}$$

Being able to write $\mathbf{B}$ as a curl makes its divergence zero automatically.

However, we've defined $\mathbf{A}$ only by specifying what its curl is, and since we need both the curl and the divergence to determine a vector field uniquely, $\mathbf{A}$ is not uniquely specified by its derivation. Since all we require is that $\mathbf{B} = \nabla\times\mathbf{A}$, we can add any vector field to $\mathbf{A}$ that has a zero curl. Since the curl of any gradient is zero, we can write the most general form of the vector potential as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' + \nabla\lambda(\mathbf{r}) \tag{12}$$

where λ is any scalar function of position. Transforming the vector potential in this way is known as a *gauge transformation*. A common choice is to choose λ so that $\nabla\cdot\mathbf{A} = 0$, which can be done by requiring

$$\nabla^2\lambda = -\frac{\mu_0}{4\pi}\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{13}$$

The quantity on the RHS is a scalar function of $\mathbf{r}$, so this is an instance of Poisson's equation. For steady currents, we can actually work out the RHS. First, note that

$$\nabla\cdot\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \mathbf{J}(\mathbf{r}')\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{14}$$

$$= -\mathbf{J}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{15}$$

where in the second line, we are now taking the derivative with respect to the primed coordinates. We now get

$$\nabla\cdot\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' = -\int_V \mathbf{J}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' \tag{16}$$

$$= -\left.\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|}\right|_\infty + \int_V \frac{\nabla'\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{17}$$

$$= 0+0 \tag{18}$$

In the second line, we've integrated by parts. The integrated term is evaluated at infinity and is zero assuming that all currents are contained within a finite volume and the second term is zero if currents are steady, since $\nabla'\cdot\mathbf{J}(\mathbf{r}')=0$. Thus $\nabla^2\lambda=0$ everywhere. This is an instance of Laplace's equation, but since it applies over all space, if we require λ to be finite at infinity, the only solution is $\lambda=$ constant, which in turn implies $\nabla\lambda=0$ so we can in fact just write

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{19}$$

With this choice of gauge, we can get another relation for $\mathbf{A}$ by using the vector identity

$$\nabla\times\mathbf{B} = \nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{20}$$

Applying $\nabla\cdot\mathbf{A}=0$ and quoting Ampère's law which says

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} \tag{21}$$

we get

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{22}$$

This is another instance of Poisson's equation, this time with a separate equation for each of the three components.

Finding the vector potential involves working out similar integrals to those for finding $\mathbf{B}$ from the Biot-Savart law.

Example 1. Suppose we have a wire segment extending from z_1 to z_2 on the z axis and carrying a steady current I (Fig. 1).

We consider a field point $\boldsymbol{r}$ which can be anywhere in space, and a source point $\boldsymbol{r}'$, which lies within the wire segment, so that

$$\boldsymbol{r}' = [0,0,z'] \tag{23}$$

The form of 19 for a linear current is then

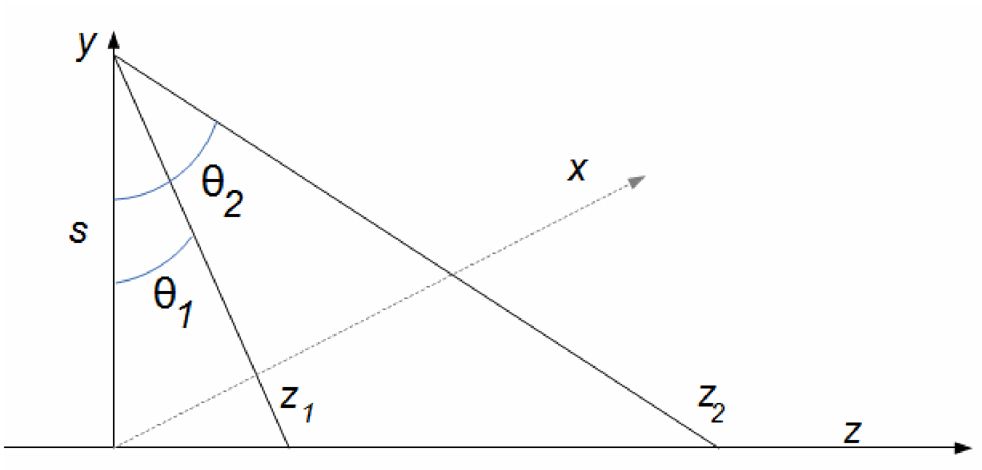

FIGURE 1. Wire segment extending from z_1 to z_2 on the z axis and carrying a steady current I. The yz plane is the plane of the page, and the x axis extends into the page.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|} \tag{24}$$

We have

$$\left|\mathbf{r}-\mathbf{r}'\right| = \sqrt{x^2+y^2+(z-z')^2} \tag{25}$$

Doing the integral with Maple we get

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}}\hat{\mathbf{z}} \tag{26}$$

$$= \frac{\mu_0 I}{4\pi}\left[-\operatorname{arcsinh}\left(\frac{z_1-z}{\sqrt{x^2+y^2}}\right)+\operatorname{arcsinh}\left(\frac{z_2-z}{\sqrt{x^2+y^2}}\right)\right]\hat{\mathbf{z}} \tag{27}$$

To check that this is correct, we can calculate (again, using Maple)

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{28}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{(z_1-z)\,y}{(x^2+y^2)^{\frac{3}{2}}\sqrt{\frac{(z_1-z)^2}{x^2+y^2}+1}} - \frac{(z_2-z)\,y}{(x^2+y^2)^{\frac{3}{2}}\sqrt{\frac{(z_2-z)^2}{x^2+y^2}+1}}\right]\hat{\boldsymbol{x}}+$$

$$\frac{\mu_0 I}{4\pi}\left[-\frac{(z_1-z)\,x}{(x^2+y^2)^{\frac{3}{2}}\sqrt{\frac{(z_1-z)^2}{x^2+y^2}+1}} + \frac{(z_2-z)\,x}{(x^2+y^2)^{\frac{3}{2}}\sqrt{\frac{(z_2-z)^2}{x^2+y^2}+1}}\right]\hat{\boldsymbol{y}} \tag{29}$$

$$=\frac{\mu_0 I}{4\pi\,(x^2+y^2)}\left\{\left[\frac{(z_1-z)\,y}{r_1^2} - \frac{(z_2-z)\,y}{r_2^2}\right]\hat{\boldsymbol{x}} + \left[-\frac{(z_1-z)\,x}{r_1^2} + \frac{(z_2-z)\,x}{r_2^2}\right]\hat{\boldsymbol{y}}\right\} \tag{30}$$

where

$$r_i \equiv \sqrt{x^2+y^2+(z-z_i)^2} \tag{31}$$

We can express this result in terms of the angles θ_1 and θ_2 as shown in Fig. 1. We need to consider a particular field point $\mathbf{r}$, so suppose we look at $\mathbf{r} = [0, s, 0]$, so $x = 0$, $y = s$, and $z = 0$ in $\boldsymbol{B}$. This gives, after simplifying

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_1}{\sqrt{s^2+z_1^2}} - \frac{z_2}{\sqrt{s^2+z_2^2}}\right]\hat{\boldsymbol{x}} \tag{32}$$

Then we can define the angles θ_i to be the angle between a line from $\mathbf{r}$ to z_i and the xy plane, for $i = 1, 2$. From the figure, we see that

$$\sin\theta_i = \frac{z_i}{\sqrt{s^2+z_i^2}} \tag{33}$$

so we have

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi s}(\sin\theta_1 - \sin\theta_2)\,\hat{\boldsymbol{x}} \tag{34}$$

If we let the two ends of the wire extend out to infinity in each direction, then $\theta_1 \to -\frac{\pi}{2}$ and $\theta_2 \to +\frac{\pi}{2}$, so we get

$$\boldsymbol{B} \to -\frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{x}} \tag{35}$$

This is the same result we would get by applying Ampère's law to a circular path of radius s around the wire.

$$\int \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I \tag{36}$$

with

$$\int \boldsymbol{B} \cdot d\boldsymbol{\ell} = 2\pi s B \tag{37}$$

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