MAGNETIZATION - BOUND CURRENTS

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By analogy with the polarization in electrostatics, we can define the magnetization $\mathbf{M}$, which is the magnetic dipole moment per unit volume. We can get a formula for the vector potential (and hence the magnetic field, although there are often easier ways of finding the field) of an object containing a given magnetization (which is a vector field). Starting with the vector potential of an ideal dipole at the origin:

$$\mathbf{A}=\frac{\mu_0}{4\pi r^2}\mathbf{m}\times\hat{\mathbf{r}} \tag{1}$$

we can write this more generally as the potential when the dipole is at position $\mathbf{r}'$:

$$\mathbf{A}=\frac{\mu_0}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{m}\times\left(\mathbf{r}-\mathbf{r}'\right) \tag{2}$$

Then if $\mathbf{M}=\mathbf{M}\left(\mathbf{r}'\right)$ we can get the potential due to a distribution of magnetic dipoles as

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{M}\left(\mathbf{r}'\right)\times\left(\mathbf{r}-\mathbf{r}'\right)d^3\mathbf{r}' \tag{3}$$

For pretty well any configuration, this integral is difficult or impossible to calculate analytically, but we can transform it into a different form, in a similar way to that used in the electrostatic case for polarization. First, we use the formula

$$\nabla'\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}=\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} \tag{4}$$

so the potential becomes

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\mathbf{M}\left(\mathbf{r}'\right)\times\nabla'\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3\mathbf{r}' \tag{5}$$

Now we can use a vector product rule:

$$\nabla\times\left(f\mathbf{V}\right)=f\left(\nabla\times\mathbf{V}\right)-\mathbf{V}\times\nabla f \tag{6}$$

With $f=1/|\mathbf{r}-\mathbf{r}'|$ and $\mathbf{V}=\mathbf{M}$ we get

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\times\mathbf{M}d^3\mathbf{r}'-\frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' \tag{7}$$

The first integral looks like the potential of a volume current density

$$\mathbf{J}_b \equiv \nabla\times\mathbf{M} \tag{8}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_b}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'-\frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' \tag{9}$$

The second integral can be transformed into a surface integral by using the divergence theorem. For a general vector field $\mathbf{V}$ and a constant vector field $\mathbf{c}$ we have, using the vector identity

$$\nabla\cdot(\mathbf{V}\times\mathbf{c})=\mathbf{V}\cdot(\nabla\times\mathbf{c})-\mathbf{c}\cdot(\nabla\times\mathbf{V}) \tag{10}$$

and the fact that $\nabla\times\boldsymbol{c}=0$ since $\boldsymbol{c}$ is constant:

$$\int\nabla\cdot(\mathbf{V}\times\mathbf{c})\,d^3\mathbf{r}'=\int\mathbf{V}\cdot(\nabla\times\mathbf{c})\,d^3\mathbf{r}'-\int\mathbf{c}\cdot(\nabla\times\mathbf{V})\,d^3\mathbf{r}' \tag{11}$$

$$=-\int\mathbf{c}\cdot(\nabla\times\mathbf{V})\,d^3\mathbf{r}' \tag{12}$$

$$=-\mathbf{c}\cdot\int(\nabla\times\mathbf{V})\,d^3\mathbf{r}' \tag{13}$$

$$=\int(\mathbf{V}\times\mathbf{c})\cdot d\mathbf{a}' \tag{14}$$

$$=\mathbf{c}\cdot\int\mathbf{V}\times d\mathbf{a}' \tag{15}$$

From Gauss's law applied to the divergence

$$\int_{\mathcal{V}}\nabla\cdot(\mathbf{V}\times\mathbf{c})\,d^3\mathbf{r}'=\int_{\mathcal{S}}(\mathbf{V}\times\mathbf{c})\cdot d\mathbf{a}' \tag{16}$$

we have

$$-\mathbf{c}\cdot\int_{\mathcal{V}}(\nabla\times\mathbf{V})\,d^3\mathbf{r}'=\int_{\mathcal{S}}(\mathbf{V}\times\mathbf{c})\cdot d\mathbf{a}' \tag{17}$$

Finally, using the triple product identity:

$$(\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A}=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \tag{18}$$

with $\boldsymbol{A}=d\boldsymbol{a}'$, $\boldsymbol{B}=\boldsymbol{V}$ and $\boldsymbol{C}=\boldsymbol{c}$, we have

$$-\mathbf{c}\cdot\int_{\mathcal{V}}(\nabla\times\mathbf{V})\,d^3\mathbf{r}'=\mathbf{c}\cdot\int_{\mathcal{S}}\mathbf{V}\times d\mathbf{a}' \tag{19}$$

Thus, since $\boldsymbol{c}$ is arbitrary:

$$-\int_{\mathcal{V}}(\nabla\times\mathbf{V})\,d^3\mathbf{r}'=\int_{\mathcal{S}}\mathbf{V}\times d\mathbf{a}' \tag{20}$$

so, substituting $\boldsymbol{V}=\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}$, 9 becomes

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'+\frac{\mu_0}{4\pi}\int_{\mathcal{S}}\frac{\mathbf{M}}{|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{a}' \tag{21}$$

If we now define a surface current

$$\mathbf{K}_b\equiv\mathbf{M}\times\hat{\mathbf{n}} \tag{22}$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface, we get

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'+\frac{\mu_0}{4\pi}\int_{\mathcal{S}}\frac{\mathbf{K}_b}{|\mathbf{r}-\mathbf{r}'|}da' \tag{23}$$

That is, we can replace the volume magnetization by a volume *bound current* $\mathbf{J}_b$ and a surface bound current $\mathbf{K}_b$ and use them to calculate the potential and the field.

In situations where there is some symmetry, we can use Ampère's law to calculate the field from these bound currents, as this usually proves a lot easier than trying to do the integrals.

Here are a few examples.

Example 1. First, suppose we have an infinite circular cylinder containing a uniform magnetization $\mathbf{M}$ parallel to its axis. Since $\mathbf{M}$ is constant, $\mathbf{J}_b=\nabla\times\boldsymbol{M}=0$ and

$$\mathbf{K}_b=M\hat{\boldsymbol{\phi}} \tag{24}$$

That is, the surface current flows in a circle around the outside of the cylinder. This is essentially the same as an infinite solenoid, so we know that the field outside the cylinder is zero, and inside, we have $B=\mu_0 nI$ where n is the number of turns per unit length and I is the current, so $nI=K_b=M$. The direction of the field is given by the right hand rule, which means it's pointing in the same direction as $\mathbf{M}$, thus inside:

$$\mathbf{B}=\mu_0\mathbf{M} \tag{25}$$

Example 2. An infinite circular cylinder of radius R has magnetization $\mathbf{M}=kr^2\hat{\boldsymbol{\phi}}$ for a constant k. The bound currents are

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = -kR^2\hat{\mathbf{z}} \tag{26}$$

$$\mathbf{J}_b = \nabla\times\mathbf{M} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\times kr^2\right)\hat{\mathbf{z}} \tag{27}$$

$$= 3kr\hat{\mathbf{z}} \tag{28}$$

Inside the cylinder, using Ampère's law the field due to the volume current (there is no contribution from the surface current) is

$$2\pi rB_J = 2\pi\mu_0(3k)\int_0^r \left(r'\right)^2 dr' \tag{29}$$

$$\mathbf{B}_J = \mu_0 kr^2\hat{\boldsymbol{\phi}} \tag{30}$$

Outside, the field due to the volume current is obtained from Ampère's law by integrating around a circle of radius r. The volume current stops at a radius $R<r$, so we have

$$\mathbf{B}_J = \mu_0 k\frac{R^3}{r}\hat{\boldsymbol{\phi}} \tag{31}$$

and the field due to the surface current is

$$2\pi rB_K = \mu_0(2\pi R)kR^2 \tag{32}$$

$$\mathbf{B}_K = -\mu_0 k\frac{R^3}{r}\hat{\boldsymbol{\phi}} \tag{33}$$

In the first line, the total surface current is obtained by multiplying K_b by the circumference of the cylinder, which is $2\pi R$. Thus the total field is

$$\mathbf{B} = \mathbf{B}_J + \mathbf{B}_K = 0 \tag{34}$$

Note that the total current (volume + surface) is zero, since they are equal in magnitude but opposite in direction. Thus by Ampère's law, the field outside the cylinder must be zero since the enclosed current is zero.

Example 3. Finally, for a cylinder with finite length and a constant **M** parallel to its axis, $\mathbf{J}_B = 0$ and $\mathbf{K}_B = M\hat{\boldsymbol{\phi}}$ as before, but $\mathbf{K}_B = 0$ on the ends of the cylinder. The magnetic field lines come out of the north pole end of the cylinder and loop around to go back into the south pole end. For a long narrow cylinder, we're back to the infinite solenoid, while for a short wide cylinder we have essentially a planar current loop.

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