MAGNETOSTATIC BOUNDARY CONDITIONS

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We've seen the behaviour of the electric field and potential at a layer of surface charge. We can do a similar analysis for the magnetic field and its behaviour as we cross a surface current.

Suppose we have a surface with a surface current **K**. At a local point, let's say this current flows in the $+x$ direction. Consider a small patch of area and build a little pillbox that straddles the surface at this area. If we make the thickness of the box infinitesimally thin, then a surface integral is essentially over the two ends of the box on either side of the current, which together we'll refer to as A. Since $\nabla\cdot\mathbf{B}=0$, Gauss's law says that

$$\int_V \nabla\cdot\mathbf{B}d^3\mathbf{r}=\int_A \mathbf{B}\cdot d\mathbf{a}=0 \tag{1}$$

Now $\mathbf{B}\cdot d\mathbf{a}=B_\perp da$ so the second integral says that

$$\int_A \mathbf{B}\cdot d\mathbf{a}=\left(B_\perp^{\text{above}}-B_\perp^{\text{below}}\right)A=0 \tag{2}$$

where the minus sign arises from the fact that the outward-pointing area element $d\boldsymbol{a}$ points in opposite directions on the two ends of the box. Therefore, the normal component of **B** is continuous across a surface current.

For the parallel component, consider first the component of **B** parallel to the surface but perpendicular to the current. We can define a little loop that straddles the surface, where the area enclosed by the loop is perpendicular to the current. If we make the vertical sides of the loop infinitesimal and the horizontal sides of length 1, then we can use Ampère's law to say

$$\oint \mathbf{B}\cdot d\mathbf{l}=B_\parallel^{\text{above}}-B_\parallel^{\text{below}}=\mu_0 I=\mu_0 K \tag{3}$$

Finally, if we take a loop perpendicular to the surface but parallel to the current, then the loop encloses zero current so $\oint \mathbf{B}\cdot d\mathbf{l}=0$ and this component of the field is continuous.

Thus the only component of **B** that has a discontinuity is the one parallel to the surface but perpendicular to the current. That is, the discontinuity is perpendicular both to the *normal* $\hat{\mathbf{n}}$ to the surface and the current $\mathbf{K}$, so the difference must be expressible as the cross product of these two vectors:

$$\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}} \tag{4}$$

To get the direction of the discontinuity, suppose we look in the $+x$ direction (that is, in the direction of the current). Then the field above the current points to the right (that is, in the $-y$ direction) and below it points to the left ($+y$ direction). Thus the difference $\mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}}$ points to the right. This works out properly if $\mathbf{K}$ points in the $+x$ direction and $\hat{\mathbf{n}}$ points in the $+z$ direction.

As for the vector potential, assuming $\nabla \cdot \mathbf{A} = 0$ means that $A_\perp$ is continuous (using the same argument as above). Since $\mathbf{B} = \nabla \times \mathbf{A}$, we can again use Stokes's theorem to integrate $\mathbf{A}$ around a loop straddling the surface:

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} \tag{5}$$

where the second integral is over the area enclosed by the loop. Unlike the integral $\oint \mathbf{B} \cdot d\mathbf{l}$ above, though, the flux of $\mathbf{B}$ enclosed by the loop diminishes to zero as we make the loop thinner and thinner. There is no infinitesimally thin sheet of magnetic field that is always enclosed by the loop as there was in the case of enclosed current. Thus in the limit, $\oint \mathbf{A} \cdot d\mathbf{l} = 0$ for any orientation of the loop across the surface, and all components of $\mathbf{A}$ are continuous across a surface current.

The derivative of $\mathbf{A}$ does have a discontinuity however. To see this, suppose we set up a coordinate system with $\hat{\mathbf{z}}$ the normal to the area patch and $\hat{\mathbf{y}}$ the direction of the current $\mathbf{K}$. Then

$$\Delta \mathbf{B} = \mathbf{B}^{\text{above}} - \mathbf{B}^{\text{below}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}} = \mu_0 K \hat{\mathbf{x}} \tag{6}$$

Writing out $\Delta \mathbf{B} = \nabla \times (\Delta \mathbf{A})$ we have

$$\Delta B_x = \partial_y (\Delta A_z) - \partial_z (\Delta A_y) = \mu_0 K \tag{7}$$

$$\Delta B_y = \partial_z (\Delta A_x) - \partial_x (\Delta A_z) = 0 \tag{8}$$

$$\Delta B_z = \partial_x (\Delta A_y) - \partial_y (\Delta A_x) = 0 \tag{9}$$

Since $\mathbf{A}$ is continuous across the surface, the derivatives in directions parallel to the surface (that is, x and y) will be the same on both sides, so x and y derivatives of $\Delta \mathbf{A}$ will all be zero. This gives us

$$\Delta B_x = \partial_z (\Delta A_y) = -\mu_0 K \tag{10}$$

$$\Delta B_y = \partial_z (\Delta A_x) = 0 \tag{11}$$

To get the third derivative we use $\nabla\cdot\mathbf{A}=0$, which gives

$$\partial_z(\Delta A_z)=-\partial_x(\Delta A_x)-\partial_y(\Delta A_y)=0 \tag{12}$$

Therefore, the y component of the normal derivative of $\mathbf{A}$ has a discontinuity:

$$\partial_z(\Delta\mathbf{A})=-\mu_0 K\hat{\mathbf{y}} \tag{13}$$

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