MAXWELL STRESS TENSOR (NON-RELATIVISTIC)

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Here we'll look at a purely classical, non-relativistic form of the tensor in electromagnetism. In doing so, we'll look only at the spatial components of the tensor, so it becomes a 3×3 matrix.

The derivation starts with a calculation of the total force due to electromagnetic fields on the charges and currents within some volume $\mathcal{V}$. From the Lorentz force law, we have

$$\mathbf{F} = \int_{\mathcal{V}}\rho\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)d^3\mathbf{r} \tag{1}$$

$$= \int_{\mathcal{V}}\left(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}\right)d^3\mathbf{r} \tag{2}$$

We can think of the integrand as a force density, or force per unit volume **f**:

$$\mathbf{f}\equiv\rho\mathbf{E}+\mathbf{J}\times\mathbf{B} \tag{3}$$

We can express this entirely in terms of fields by using a couple of Maxwell's equations:

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} \tag{4}$$

$$\mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{5}$$

So we get

$$\mathbf{f} = \left(\epsilon_0\nabla\cdot\mathbf{E}\right)\mathbf{E}+\left[\frac{1}{\mu_0}\nabla\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right]\times\mathbf{B} \tag{6}$$

We now need to do a bit of vector calculus gymnastics. From the product rule

$$\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}+\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t} \tag{7}$$

and from Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{8}$$

Combining these two we get

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \tag{9}$$

$$= \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{10}$$

We can insert this into 6 and while we're at it, we can add on a term $\frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B}$. This is always zero because $\nabla \cdot \mathbf{B} = 0$, but it gives the equation a symmetry that will be useful in a minute. We get for the force density:

$$\mathbf{f} = \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \tag{11}$$

$$= \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} - \tag{12}$$

$$\epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \epsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E})$$

Now another identity from vector calculus says

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} \tag{13}$$

If $\mathbf{A} = \mathbf{B} = \mathbf{E}$, we get

$$\nabla \left(E^2\right) = 2\mathbf{E} \times (\nabla \times \mathbf{E}) + 2 (\mathbf{E} \cdot \nabla) \mathbf{E} \tag{14}$$

so

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2} \nabla \left(E^2\right) - (\mathbf{E} \cdot \nabla) \mathbf{E} \tag{15}$$

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2} \nabla \left(B^2\right) - (\mathbf{B} \cdot \nabla) \mathbf{B} \tag{16}$$

Putting this into 12 we get

$$\mathbf{f} = \epsilon_0 (\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\cdot\mathbf{B})\mathbf{B} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) - \frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\times(\nabla\times\mathbf{E}) \tag{17}$$

$$= \epsilon_0 (\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\cdot\mathbf{B})\mathbf{B} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) + \epsilon_0(\mathbf{E}\cdot\nabla)\mathbf{E} + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{18}$$

$$= \epsilon_0\left[(\nabla\cdot\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\mathbf{B})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B}\right] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) \tag{19}$$

It might not seem that we're making any progress, since the equations just get longer with each alteration. However, we can now introduce the Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}$ which is a 3×3 matrix with components defined by

$$\boxed{T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)} \tag{20}$$

Note that the tensor is symmetric: $T_{ij} = T_{ji}$. If we define the scalar product of the tensor with an ordinary vector to be another vector:

$$\left[\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}}\right]_j = \sum_i a_iT_{ij} \tag{21}$$

where the subscript j indicates the jth component of the resulting vector, then the divergence is

$$\left[\nabla\cdot\overleftrightarrow{\mathbf{T}}\right]_j = \sum_i \partial_i T_{ij} \tag{22}$$

$$= \epsilon_0 \sum_i \left((\partial_i E_i) E_j + E_i (\partial_i E_j) - \frac{1}{2}\delta_{ij}\partial_i E^2 \right) + \tag{23}$$

$$\frac{1}{\mu_0}\sum_i \left((\partial_i B_i) B_j + B_i (\partial_i B_j) - \frac{1}{2}\delta_{ij}\partial_i B^2 \right)$$

$$= \epsilon_0 \left((\nabla\cdot\mathbf{E}) E_j + (\mathbf{E}\cdot\nabla) E_j - \frac{1}{2}\partial_j E^2 \right) + \tag{24}$$

$$\frac{1}{\mu_0}\left((\nabla\cdot\mathbf{B}) B_j + (\mathbf{B}\cdot\nabla) B_j - \frac{1}{2}\partial_j B^2 \right)$$

Comparing this with 19, we see that we can write $\mathbf{f}$ in terms of $\overleftrightarrow{\mathbf{T}}$ and the Poynting vector as

$$\boxed{\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}} \tag{25}$$

The total force on the volume is then

$$\mathbf{F} = \int_{\mathcal{V}} \mathbf{f} d^3\mathbf{r} \tag{26}$$

$$= \int_{\mathcal{V}} \left(\nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t} \right) d^3\mathbf{r} \tag{27}$$

From the formula 22 for the divergence, we can see that the vector resulting from the divergence has as its components the divergences of each column of $\overleftrightarrow{\mathbf{T}}$. Therefore we can apply the divergence theorem to the first term in the integrand to get

$$\boxed{\mathbf{F} = \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \epsilon_0\mu_0\frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{S} d^3\mathbf{r}} \tag{28}$$

where $\mathcal{S}$ is any surface that encloses only the charges and currents within $\mathcal{V}$.

Example 1. We consider the magnetic force between the two halves of a spherical shell of radius R and surface charge density σ rotating with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$.

In Example 5.11 in Griffiths's book, he finds the vector potential to be

$$\mathbf{A}(r,\theta,\phi)=\begin{cases}\frac{1}{3}\mu_0 R\omega\sigma r\sin\theta\hat{\boldsymbol{\phi}} & r\le R\\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}} & r\ge R\end{cases}\tag{29}$$

Here, R is the radius of the shell, σ is the surface charge density and ω is the angular velocity, where the sphere's axis is taken to be the z axis.

From this we can calculate the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ using the standard formula for the curl in spherical coordinates:

$$\mathbf{B}=\begin{cases}\frac{2\mu_0 R\omega\sigma}{3}\left(\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}\right) & r\le R\\ \frac{\mu_0 R^4\omega\sigma}{3}\frac{1}{r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right) & r\ge R\end{cases}\tag{30}$$

Using the stress tensor, we can integrate over *any* volume that encloses the upper half of the sphere, so we can choose the half space consisting of all space above the xy plane (we're assuming that the centre of the sphere is at the origin, so the xy plane contains the sphere's equator). Since the distribution of charges and currents is finite, all fields will go to zero at infinity, so we need to integrate only over the xy plane.

In the xy plane, $\theta=\frac{\pi}{2}$ and in this plane, the unit vector $\hat{\boldsymbol{\theta}}$ points in the $-z$ direction, so is equivalent to the unit vector $-\hat{\mathbf{z}}$. Thus we have

$$\mathbf{B}=\begin{cases}\frac{2\mu_0 R\omega\sigma}{3}\hat{\mathbf{z}} & r<R\\ -\frac{\mu_0 R^4\omega\sigma}{3}\frac{1}{r^3}\hat{\mathbf{z}} & r>R\end{cases}\tag{31}$$

Since we're interested only in the magnetic field, we can ignore $\mathbf{E}$ here, although there is a repulsive force between the two hemispheres due to the electric field as well. Also, as the currents are steady, $\partial\mathbf{S}/\partial t=0$.

From the symmetry of the problem, the force is in the z direction, so we need to work out only $\left[\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}\right]_z$. We get $T_{xz}=T_{yz}=0$ because $B_x=B_y=0$ on the xy plane, so we're left with just T_{zz} in 20:

$$T_{zz}=\frac{1}{2\mu_0}B_z^2=\begin{cases}\frac{2}{9}\mu_0\sigma^2\omega^2R^2 & r<R\\ \frac{1}{18}\mu_0\sigma^2\omega^2\frac{R^8}{r^6} & r>R\end{cases}\tag{32}$$

The total force is then (the minus sign is because $T_{zz}>0$ and $d\mathbf{a}$ points towards $-z$):

$$\mathbf{F} = \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \tag{33}$$

$$= -\hat{\mathbf{z}} \left[\frac{2}{9} \mu_0 \sigma^2 \omega^2 R^2 2\pi \int_0^R r\, dr + \frac{2\pi}{18} \mu_0 \sigma^2 \omega^2 R^8 \int_R^\infty \frac{r\, dr}{r^6} \right] \tag{34}$$

$$= -\hat{\mathbf{z}} \left(\frac{2\pi}{9} \mu_0 \sigma^2 \omega^2 R^4 + \frac{\pi}{36} \mu_0 \sigma^2 \omega^2 R^4 \right) \tag{35}$$

$$= -\frac{\pi}{4} \mu_0 \sigma^2 \omega^2 R^4 \hat{\mathbf{z}} \tag{36}$$

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