MAXWELL'S CORRECTION TO AMPÈRE'S LAW - DISPLACEMENT CURRENT

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Ampère's law gives the magnetic field generated by a steady current. We run into problems if we try to apply it to a non-static situation. Ampère's law in differential form is

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \tag{1}$$

Since the divergence of a curl is always zero, this law implies that

$$\nabla\cdot(\nabla\times\mathbf{B})=0=\mu_0\nabla\cdot\mathbf{J} \tag{2}$$

For steady currents, the RHS is indeed true, but in general

$$\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t} \tag{3}$$

where ρ is the charge density. Thus for time-varying currents, Ampère's law is not true. Maxwell fixed this problem by making a postulate that is, in a way, the complement of Faraday's postulate that a changing electric field produces a magnetic field; Maxwell proposed that a changing electric field induces a magnetic field. In particular, he proposed that

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

This fixes the divergence problem, since we now get

$$\nabla\cdot(\nabla\times\mathbf{B})=0=\mu_0\nabla\cdot\mathbf{J}+\mu_0\epsilon_0\frac{\partial(\nabla\cdot\mathbf{E})}{\partial t} \tag{5}$$

but from Gauss's law, $\nabla\cdot\mathbf{E}=\rho/\epsilon_0$, so, using 3, the RHS is indeed now zero.

Since $\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$ has the dimensions of current density, Maxwell called it the *displacement current:*

$$\mathbf{J}_d\equiv\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{6}$$

Example 1. We have a circular solid wire of radius a that has a narrow gap of width $w \ll a$ cut into it. If a steady current I is passed along the wire, the two surfaces on either side of the gap act as a parallel plate capacitor, as charge builds up on these surfaces. As a result, the electric field between the plates changes as the charge builds up, so we should get an induced magnetic field. Neglecting fringe effects at the edges of the plates, the electric field between the plates is $E = \sigma/\epsilon_0$ where σ is the surface charge density. What we're really interested in isn't E itself, but its derivative, which we can write as

$$\begin{aligned}
\frac{\partial E}{\partial t} &= \frac{1}{\epsilon_0}\frac{d\sigma}{dt} &(7)\\
&= \frac{1}{\pi a^2 \epsilon_0}\frac{dQ}{dt} &(8)\\
&= \frac{I}{\pi a^2 \epsilon_0} &(9)
\end{aligned}$$

where Q is the total charge on one side of the gap (with $-Q$ on the other side).

Taking a circular path of radius r between the plates, there is no current here so $\mathbf{J} = 0$ and we have

$$\begin{aligned}
\oint \mathbf{B} \cdot d\boldsymbol{\ell} &= 2\pi r B &(10)\\
&= \mu_0 \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} &(11)\\
&= \mu_0 \epsilon_0 \pi r^2 \frac{I}{\pi a^2 \epsilon_0} &(12)\\
&= \mu_0 I \frac{r^2}{a^2} &(13)\\
B &= \mu_0 \frac{Ir}{2\pi a^2} &(14)
\end{aligned}$$

By symmetry and the right hand rule, if we take the z direction to be from the positive to the negative plate, then

$$\mathbf{B} = \mu_0 \frac{Ir}{2\pi a^2} \hat{\boldsymbol{\phi}} \tag{15}$$

Thus a circumferential magnetic field appears in the gap, whose strength is proportional to the distance r from the axis of the wire.

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