MAXWELL'S EQUATIONS IN MATTER

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Inside matter, we've seen that the polarization and magnetization give rise to bound charges and bound currents. The earlier results applied in the electrostatic and magnetostatic cases, respectively, so we'd like to generalize these to get the corresponding equations in electrodynamics.

For a medium with polarization **P** and magnetization **M**, the static cases are

$$-\nabla\cdot\mathbf{P} = \rho_b \tag{1}$$
$$\mathbf{P}\cdot\hat{\mathbf{n}} = \sigma_b \tag{2}$$
$$\nabla\times\mathbf{M} = \mathbf{J}_b \tag{3}$$
$$\mathbf{M}\times\hat{\mathbf{n}} = \mathbf{K}_b \tag{4}$$

where ρ_b and σ_b are the bound volume and surface charge densities and $\mathbf{J}_b$ and $\mathbf{K}_b$ are the bound volume and surface currents. These equations assume that nothing changes with time.

Now suppose that the polarization does change with time, so that $\mathbf{P} = \mathbf{P}(t)$. Considering a small cylindrical volume element with an axis parallel to **P**, there is no bound surface charge on the sides of the cylinder, and on one end $\sigma_b = P$ and on the other $\sigma_b = -P$. If P increases by ΔP over time Δt then the surface charge density also increases (or decreases, at the negative end, but both ends increase in magnitude) by ΔP. This change in charge density must be due to a current density $\mathbf{J}_p$ parallel to **P** of amount

$$\mathbf{J}_p = \frac{\partial\sigma_b}{\partial t}\hat{\mathbf{n}} = \frac{\partial\mathbf{P}}{\partial t} \tag{5}$$

This current density is called the *polarization current*. It is a current that is not present at all in the static case, so it's a new current in addition to the bound current $\mathbf{J}_b$ and free current $\mathbf{J}_f$.

If the magnetization changes with time, this will change the bound currents $\mathbf{J}_b$ and $\mathbf{K}_b$ but it doesn't introduce any new currents into the system, so the only change we need to make in a dynamic system is the addition of the polarization current $\mathbf{J}_p$. The charge density ρ is thus still composed of two contributions:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P} \tag{6}$$

but the current density now has an additional term:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \tag{7}$$

Gauss's law in matter can thus be written

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0}\left(\rho_f - \nabla \cdot \mathbf{P}\right) && (8)\\
\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) &= \rho_f && (9)\\
\nabla \cdot \mathbf{D} &= \rho_f && (10)
\end{aligned}$$

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ is the displacement.

Ampère's law with Maxwell's correction must take into account the polarization current, so we have

$$\begin{aligned}
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && (11)\\
&= \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}\right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && (12)\\
\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) &= \mathbf{J}_f + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E} + \mathbf{P}) && (13)\\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} && (14)
\end{aligned}$$

where $\boldsymbol{H}$ is the auxiliary magnetic field.

The other two Maxwell equations don't change, since they don't refer explicitly to charge or current. Thus we get Maxwell's equations in matter:

$$\boxed{\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_f\\
\nabla \cdot \mathbf{B} &= 0\\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}} \tag{15}$$

For linear, homogeneous materials, the polarization and magnetization depend linearly on $\mathbf{E}$ and $\mathbf{B}$, respectively, so

$$\mathbf{D} = \epsilon\mathbf{E} \tag{16}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} \tag{17}$$

where

$$\epsilon = \epsilon_0 (1+\chi_e) \tag{18}$$

$$\mu = \mu_0 (1+\chi_m) \tag{19}$$

are the permittivity and permeability of the material. Maxwell's original displacement current, $\epsilon_0 \partial\mathbf{E}/\partial t$, becomes $\partial\mathbf{D}/\partial t$.

Example 1. Suppose we have a large parallel plate capacitor embedded in sea water and maintain an AC voltage across the plates, so that

$$V(t) = V_0 \cos(2\pi\nu t) \tag{20}$$

In terms of the conductivity σ of the sea water, the current density due to conduction is $\mathbf{J} = \sigma\mathbf{E}$ and for a capacitor with plates separated by a distance d, $E(t) = V(t)/d$, so

$$J = \frac{\sigma V_0}{d}\cos(2\pi\nu t) \tag{21}$$

The displacement current is

$$J_d = \frac{\partial D}{\partial t} \tag{22}$$

$$= \epsilon\frac{\partial E}{\partial t} \tag{23}$$

$$= -2\pi\epsilon\nu\frac{V_0}{d}\sin(2\pi\nu t) \tag{24}$$

The ratio of the amplitudes is therefore

$$\frac{J}{J_d} = \frac{\sigma}{2\pi\epsilon\nu} = \frac{1}{2\pi\epsilon\nu\rho} \tag{25}$$

where ρ is the resistivity of sea water.

Plugging in the values given by Griffiths in problem 7.37, we have $\rho = 0.23\ \Omega\cdot\mathrm{m}$, $\epsilon = 81\epsilon_0 = 7.17\times10^{-10}\ \mathrm{m}^{-3}\mathrm{kg}^{-1}\mathrm{s}^4\mathrm{A}^2$, $\nu = 4\times10^8\ \mathrm{s}^{-1}$ and

$$\frac{J}{J_d} = 2.41 \tag{26}$$

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