MAXWELL'S EQUATIONS

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Post date: 28 Mar 2021.

We can now summarize Maxwell's equations

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{2}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{4}$$

Equation 1 is Gauss's law from electrostatics, and relates the electric field $\boldsymbol{E}$ to the charge density ρ. 2 is derived from the Biot-Savart law, and expresses the experimental fact that there are no magnetic monopoles. 3 is Faraday's law, and shows that a changing magnetic field induces an electric field. 4 is a combination of Ampère's law for steady currents with Maxwell's correction to Ampère's law, in which a changing electric field induces a magnetic field. These four equations (and their variants for use in dielectrics) are the basis of all classical electromagnetism. From them, we can derive all the laws governing electric circuits, and further, the existence of electromagnetic radiation in the form of waves. They must rank as one of the towering achievements of 19th century physics.

Example 1. A novel example of electric and magnetic fields that satisfy these equations is

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}H(vt-r)\hat{\mathbf{r}} \tag{5}$$

$$\mathbf{B} = 0 \tag{6}$$

where $H(x)$ is the step function

$$H(x)\equiv\begin{cases}1 & x>0\\ 0 & x<0\end{cases} \tag{7}$$

and the equations assume spherical coordinates.

Since $B = 0$, $\nabla\cdot\mathbf{B} = \nabla\times\mathbf{B} = 0$ and, since $\mathbf{E}$ has only a radial component which depends only on r, $\nabla\times\mathbf{E} = 0$ (if you look up the form of the curl in spherical coordinates, the only derivatives of E_r are with respect to θ and ϕ which are both zero).

You might think that $\nabla\cdot\mathbf{E}$ is fairly easy to calculate using the definition of the divergence in spherical coordinates, but we must remember that something bizarre happens at $r = 0$; we must use the formula for the 3-d delta function:

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}) \tag{8}$$

and the product rule for the divergence of the product of a scalar field f and a vector field $\mathbf{A}$:

$$\nabla\cdot(f\mathbf{A}) = \mathbf{A}\cdot\nabla f + f\nabla\cdot\mathbf{A} \tag{9}$$

Here we have

$$f = H(vt-r) \tag{10}$$

$$\mathbf{A} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \tag{11}$$

Using the derivative of the step function $dH(x)/dx = \delta(x)$ we have

$$\frac{dH(vt-r)}{dr} = -\delta(vt-r) \tag{12}$$

so

$$\nabla\cdot\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot(\nabla H(vt-r)) - H(vt-r)\frac{q}{4\pi\epsilon_0}\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) \tag{13}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot(\delta(vt-r)\hat{\mathbf{r}}) - H(vt-r)\frac{q}{\epsilon_0}\delta^3(\mathbf{r}) \tag{14}$$

The second term is non-zero only at $\mathbf{r} = 0$ so, assuming $vt > 0$ we can replace the step function by 1, since at $\boldsymbol{r} = 0$ we have

$$H(vt-r) = H(vt) = +1 \tag{15}$$

We therefore have

$$\nabla\cdot\mathbf{E} = \frac{q}{4\pi r^2\epsilon_0}\delta(vt-r) - \frac{q}{\epsilon_0}\delta^3(\mathbf{r}) \tag{16}$$

The charge density is therefore

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} \tag{17}$$

$$= \frac{q}{4\pi r^2}\delta(vt-r) - q\delta^3(\mathbf{r}) \tag{18}$$

This corresponds to a point charge $-q$ at the origin and an expanding (with radial speed v) spherical shell of charge of total amount $+q$. The electric field inside the shell is due entirely to the point charge at the origin, and the field outside the shell is zero since the total enclosed charge is zero.

The current density is obtained from $\nabla \times \mathbf{B} = 0$, so:

$$\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = 0 \tag{19}$$

$$\mathbf{J} = -\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{20}$$

$$= \frac{qv}{4\pi r^2}\delta(vt-r)\,\hat{\mathbf{r}} \tag{21}$$

The expanding shell provides the current density; the stationary charge at the origin doesn't contribute.

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