MULTIPOLE EXPANSION IN ELECTROSTATICS

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For a given charge distribution, we can write down a *multipole expansion*, which gives the potential as a series in powers of $1/r$, where r is the distance from the origin to the observation point.

We know that the potential in general is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{r}')\frac{d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

In the integral, $\mathbf{r}'$ is the position of charge element $\rho(\mathbf{r}')d^3\mathbf{r}'$. From the law of cosines

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+r'^2-2rr'\cos\theta'} \tag{2}$$

where θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$. We can rewrite this as

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} \tag{3}$$

$$= \frac{1}{r}\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta'}} \tag{4}$$

From the theory of Legendre polynomials, it is known that the last factor in this expression is a *generating function* for the polynomials. That is, if we write the square root as an power series, we get

$$\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta'}} = \sum_{n=0}^{\infty}P_n(\cos\theta')\left(\frac{r'}{r}\right)^n \tag{5}$$

The coefficient of $\left(\frac{r'}{r}\right)^n$ in the series is the Legendre polynomial $P_n(\cos\theta')$. This can be verified for the first few terms by calculating the Taylor series expansion of the square root term about $r'/r=0$. This is tedious to do by hand, but using Maple, we get, defining $s\equiv r'/r$:

$$\frac{1}{\sqrt{1+s^2-2s\cos\theta'}} = 1+s\cos\theta'+s^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)+s^3\left(\frac{5}{2}\cos^3\theta'-\frac{3}{2}\cos\theta'\right)+\ldots \tag{6}$$

We can compare the coefficients of the powers of s with tables of the Legendre polynomials to verify that that's what they are.

It is important to note that the angle θ' is equivalent to the angle θ in spherical coordinates *only* if the observation point $\mathbf{r}$ lies on the z axis, since that is the only configuration where the angle between the observation vector and a charge element corresponds to the spherical coordinate angle θ. (A more general multipole expansion uses spherical harmonics rather than just Legendre polynomials, but that's a topic for a more advanced post.)

With this restriction, we can substitute the series expansion back into 1 to get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}\int\rho(\mathbf{r}')\sum_{n=0}^{\infty}P_n(\cos\theta')\left(\frac{r'}{r}\right)^n d^3\mathbf{r}' \tag{7}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int\rho(\mathbf{r}')P_n(\cos\theta')r'^n d^3\mathbf{r}' \tag{8}$$

The first few terms in this series have special names. The $n=0$ term is

$$\frac{1}{4\pi\epsilon_0 r}\int\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{Q}{4\pi\epsilon_0 r} \tag{9}$$

where Q is the total charge. This is called the *monopole* term, and shows that to a first approximation, the potential of any charge distribution is just the potential of a point charge with the same total charge.

The next term in the series is

$$\frac{1}{4\pi\epsilon_0 r^2}\int r'P_1(\cos\theta')\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^2}\int r'\cos\theta'\rho(\mathbf{r}')d^3\mathbf{r}' \tag{10}$$

This is called the *dipole* term. We can compare this with our more general analysis of the electric dipole, which gives the potential as

$$V(\boldsymbol{r}) = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3} \tag{11}$$

where $\boldsymbol{p}$ is the dipole moment, given by

$$\boldsymbol{p} = qd\hat{\mathbf{z}} \tag{12}$$

Here, the dipole consists of two equal but opposite charges $\pm q$ situated at $z = +\frac{d}{2}$ for $+q$ and $z = -\frac{d}{2}$ for $-q$. Since the analysis above applies only to cases where $\boldsymbol{r}$ is parallel to $\hat{\boldsymbol{z}}$, we have $\theta' = 0$ so

$$\boldsymbol{p} \cdot \boldsymbol{r} = qdr \tag{13}$$

and the potential along the z axis becomes

$$V(\boldsymbol{r}) = \frac{qd}{4\pi\epsilon_0 r^2} \tag{14}$$

We can write the charge density for the two point charges in the dipole as

$$\rho\left(\boldsymbol{r}'\right) = q\delta\left(r' - \frac{d}{2}\right) - q\delta\left(r' + \frac{d}{2}\right) \tag{15}$$

so the integral in 10 becomes

$$\int r' \cos\theta' \rho(\mathbf{r}') d^3\mathbf{r}' = q \int r' \left[\delta\left(r' - \frac{d}{2}\right) - \delta\left(r' + \frac{d}{2}\right)\right] dr' \tag{16}$$

$$= q\left(\frac{d}{2} - \left(-\frac{d}{2}\right)\right) \tag{17}$$

$$= qd \tag{18}$$

With this result, 10 reduces to 14.

Returning to 8, for $n = 2$, we get the *quadrupole* term

$$\frac{1}{4\pi\epsilon_0 r^3} \int r'^2 P_2(\cos\theta') \rho(\mathbf{r}') d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^3} \int r'^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) \rho(\mathbf{r}') d^3\mathbf{r}' \tag{19}$$

Finally, for $n = 3$ we get the *octopole* term

$$\frac{1}{4\pi\epsilon_0 r^4} \int r'^3 P_3(\cos\theta') \rho(\mathbf{r}') d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^4} \int r'^3 \left(\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta'\right) \rho(\mathbf{r}') d^3\mathbf{r}' \tag{20}$$

As an example, consider a solid sphere with a charge density

$$\rho(\mathbf{r}') = k\frac{R}{r'^2}(R - 2r')\sin\theta' \tag{21}$$

We can use the integrals above to find the first non-zero term in the series, and thus get an approximation for the potential. *Note that we can do this only for points on the z axis.*

By direct calculation, we have for the monopole term 9:

$$\frac{1}{4\pi\epsilon_0 r}\int\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r}\int_0^R\int_0^\pi\int_0^{2\pi}k\frac{R}{r'^2}(R-2r')\sin\theta' r'^2\sin\theta' d\phi' d\theta' dr' \tag{22}$$

$$= \frac{kR}{4\pi\epsilon_0 r}\int_0^R\int_0^\pi\int_0^{2\pi}(R-2r')\sin^2\theta' d\phi' d\theta' dr' \tag{23}$$

$$= 0 \tag{24}$$

since the integral over r' gives zero. Thus the monopole term vanishes, as it always does if the total charge is zero.

For the dipole term, we get from 10

$$\frac{1}{4\pi\epsilon_0 r^2}\int r'\cos\theta'\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^2}\int_0^R\int_0^\pi\int_0^{2\pi}k\frac{R}{r'^2}(R-2r')r'^3\sin\theta'\cos\theta'\sin\theta' d\phi' d\theta' dr' \tag{25}$$

$$= 0 \tag{26}$$

This time, the integral over θ' gives zero, since the term $\cos\theta'\sin^2\theta'$ is odd relative to the interval $[0,\pi]$.

For the quadrupole term, using Maple for the integral, we have from 19

$$\frac{1}{4\pi\epsilon_0 r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d^3\mathbf{r}' = \frac{1}{4\pi\epsilon_0 r^3}\int_0^R\int_0^\pi\int_0^{2\pi}k\frac{R}{r'^2}(R-2r')\sin\theta' \times r'^4\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\sin\theta' d\phi' d\theta' dr' \tag{27}$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left(\frac{\pi^2 kR^5}{48}\right) \tag{28}$$

$$= \frac{\pi kR^5}{192\epsilon_0 r^3} \tag{29}$$

The octopole term 20 comes out to zero, since the terms in θ' are again odd relative to the interval $[0,\pi]$. Thus the quadrupole term is the only non-zero term in the first 4 terms in the multipole expansion.

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