OHM'S LAW IN THE MICROSCOPIC REALM

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Ohm's law, which is a staple of introductory physics courses, is an empirical relationship between the voltage V applied to a substance and the resulting current I that flows through it. V and I are related by a property known as the resistance R, giving the relation

$$V = IR \tag{1}$$

It should be empasized that Ohm's law is not derivable from the principles of electromagnetism; rather it is a relation that is observed experimentally, and doesn't apply to every substance under all conditions. The resistance has units of ohms (symbol Ω), which is equivalent to 1 volt per ampere.

The resistance depends not only on the substance but also on the geometry of the sample. However, it is known that for samples of the same size and shape, the resistance can vary greatly depending on the substance from which the sample is made. Most metals have low resistances, while substances such as glass or rubber have very high resistances. A few substances, such as silicon, have intermediate resistances.

There are a couple of properties that can be used to characterize a substance's resistance behaviour that are independent of the size and shape of the sample. We can define a property ρ called the *resistivity* or *specific resistance* as follows. We consider a sample of uniform cross section, with length ℓ and cross-sectional area S. Then we have

$$R = \rho\frac{\ell}{S} \tag{2}$$

The most common shape, at least for conductors, is a wire.

From its definition, ρ has units of ohm $\cdot$ m, so its numerical value is the resistance of a sample with a cross-sectional area of 1 metre^2 and a length of 1 metre, which is a very big wire! The best conductor is silver, with $\rho = 1.62\times10^{-8}$ Ωm. If we have a silver wire 10 cm long with a cross-sectional area of 1 mm^2, its resistance would be

$$R = 1.62\times10^{-8}\frac{0.1}{10^{-6}} = 1.62\times10^{-3}\Omega \tag{3}$$

Typical insulators have resistivities in the range $10^{12}-10^{15}\Omega$m. Glass, for example, has $\rho > 10^{15}$, so a thread of glass (as in a fibre optic cable, for example) with the same dimensions as the silver wire above would have a resistance of around $10^{24}\Omega$.

The reciprocal of resistivity is called the *electric conductivity* (symbol $\sigma \equiv 1/\rho$) and the reciprocal of resistance is called the *conductance* (symbol $G \equiv 1/R$).

The reason that Ohm's law isn't derivable from basic electromagnetism is that it's not entirely an electromagnetic effect. In a conductor, applying an electric field causes free charges (usually electrons) to move, but due to the structure of the conductor, they will not be able to travel very far before colliding with an atom. If an electron with charge $-e$ is placed in an electric field E, it feels a force $-eE$, so by Newton's law

$$m\frac{dv}{dt} = -eE \tag{4}$$

If the electron is free (that is, it never collides with anything), this naive equation would give the electron a constant acceleration which means it would just get faster and faster. Since this doesn't happen, we need to modify 4 to account for collisions. The standard way this is done is by introducing a viscous force on the electron. Without getting into the theory of fluid flow, we can create a simple model by assuming that the viscous force on an electron is opposite to its direction of motion and proportional to its speed. This makes a certain qualitative sense, as the faster an electron moves, the more likely it is to collide with an atom.

The viscosity is measured by a *coefficient of viscosity* η which again is not easily derivable from first principles; it's another quantity that is measured experimentally. The modified form of 4 is then

$$m\frac{dv}{dt} = -eE - \eta v \tag{5}$$

We can solve this ODE either using Maple, or by hand. For the latter, we have

$$\frac{dv}{eE+\eta v} = -\frac{dt}{m} \tag{6}$$

Integrating both sides gives

$$\frac{1}{\eta}\ln(eE+\eta v) = -\frac{t}{m} + \ln C \tag{7}$$

where C is a constant. We can multiply through by η and then exponentiate both sides to get

$$eE + \eta v = Ce^{-\eta t/m} \tag{8}$$

or

$$v = \frac{C}{\eta} e^{-\eta t/m} - \frac{eE}{\eta} \tag{9}$$

If $v(0) = 0$, then

$$C = eE \tag{10}$$

and we have

$$v(t) = \frac{eE}{\eta} \left(e^{-\eta t/m} - 1 \right) \tag{11}$$

In this model, the velocity contains an exponentially decaying term, and after a time

$$t \approx \frac{m}{\eta} \tag{12}$$

the velocity tends to a steady state of

$$v(t) \to -\frac{eE}{\eta} \tag{13}$$

How long does it take for the speed to settle down to its asymptotic value? We can get an idea as follows. Suppose the number density of free electrons is n. Then the current density (current per unit cross-sectional area) is

$$i = -env \tag{14}$$

Using 13 as an estimate of the velocity, we have

$$i \to \frac{e^2 n}{\eta} E \tag{15}$$

or

$$E \to \frac{\eta}{e^2 n} i \tag{16}$$

$$= \frac{\eta}{e^2 n} \frac{I}{S} \tag{17}$$

where S is the total cross-sectional area of the current. The electric field E is in volts per metre, so for a conductor of length ℓ, the total voltage V along the conductor is $V = E\ell$ and we have

$$V \to \frac{\eta}{e^2 n} \frac{I\ell}{S} \tag{18}$$

From Ohm's law 1, we have, at the steady state

$$V = RI = \frac{\eta}{e^2 n}\frac{\ell}{S} I \tag{19}$$

so, using 2

$$R = \frac{\eta}{e^2 n}\frac{\ell}{S} = \rho\frac{\ell}{S} \tag{20}$$

The decay time 12 is therefore

$$t \approx \frac{m}{\eta} = \frac{m}{ne^2\rho} \tag{21}$$

To use this, we need some estimates for the values on the RHS. We know that, for electrons

$$\begin{aligned} m &= 9.1 \times 10^{-31}\text{kg} \\ e &= 1.6 \times 10^{-19}\text{C} \end{aligned} \tag{22}$$

For silver

$$\rho = 1.62 \times 10^{-8}\ \Omega\text{m} \tag{23}$$

To get n, we use the density of silver (about $10.5 \times 10^3\text{kg m}^{-3}$) and the mass of a typical silver atom (1.79×10^{-25}kg) which gives about 6×10^{28} atoms per m^3. I'm not sure how many free electrons there are per atom, but assuming there is only one, we have

$$n = 6 \times 10^{28}\text{m}^{-3} \tag{24}$$

Plugging all these into 21 we get

$$t \approx 3.6 \times 10^{-14}\text{s} \tag{25}$$

It's highly unlikely that any measurement of current could detect changes in this time range, so essentially Ohm's law is valid from the time the switch is thrown to turn the current on.