POLARIZED OBJECTS IN ELECTROSTATICS

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The potential of an ideal dipole is

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}\cdot\mathbf{p} \tag{1}$$

where $\mathbf{r}$ is the vector *from* the dipole *to* the observation point.

If we now consider an object consisting of polarized dielectric, we can define the *polarization density*, or polarization per unit volume as $\mathbf{P}$. The polarization density $\boldsymbol{P}(\boldsymbol{r}')$ can be regarded as the dipole moment of an infinistesimal volume around $\boldsymbol{r}'$. The potential due to such an object is then

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{2}$$

where $\mathcal{V}$ is the volume of the object. Note that this is a volume integral over the *primed* coordinates $\mathbf{r}'$, that is, over the location of the volume element containing the polarized material.

We can transform this integral by doing a calculation of a gradient:

$$\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=\frac{\partial}{\partial x'}\left[\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}\right]\hat{\mathbf{x}}+\ldots \tag{3}$$

$$=\frac{(x-x')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\hat{\mathbf{x}}+\ldots \tag{4}$$

$$=\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{5}$$

where we've omitted the terms in $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ in the first two lines since they have the same form. We can therefore write the original integral as

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\mathbf{P}\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' \tag{6}$$

A standard theorem from vector calculus says, for a function f and vector field $\mathbf{A}$:

$$\nabla(f\mathbf{A}) = \mathbf{A}\cdot\nabla f + f\nabla\cdot\mathbf{A} \tag{7}$$

so we can transform the integral to get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\nabla'\cdot\left(\frac{\mathbf{P}}{|\mathbf{r}-\mathbf{r}'|}\right)d^3\mathbf{r}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{P}d^3\mathbf{r}' \tag{8}$$

The first integral is a volume integral of a divergence so we can apply the divergence theorem to transform this to a surface integral, so we finally get

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{S}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{P}d^3\mathbf{r}' \tag{9}$$

where the first integral is over the surface of the polarized object, and the second integral is over its volume. That is, the potential of a polarized object can be expressed as the sum of the potential of a surface charge density and a volume charge density, where we have

$$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} \tag{10}$$

$$\rho_b \equiv -\nabla'\cdot\mathbf{P} \tag{11}$$

These charge distributions are known as *bound charges*, which is why we've used a subscript b. If we're interested in the electric field of a dipole distribution, we can work out these integrals and then take the negative gradient to get $\mathbf{E} = -\nabla V$, or, if the problem has the right symmetry, we can use Gauss's law to work out the fields of the two charge distributions and then add them together.

Example 1. Suppose we have a sphere of radius R with a polarization density given by

$$\mathbf{P}(\mathbf{r}') = k\mathbf{r}' \tag{12}$$

where k is a constant. Then

$$\sigma_b = kR \tag{13}$$

$$\rho_b = -k\nabla'\cdot\mathbf{r}' \tag{14}$$

$$= -3k \tag{15}$$

where the last line can be found by expressing $\mathbf{r}'$ in rectangular coordinates, so we have

$$\nabla' \cdot \boldsymbol{r}' = \nabla' \cdot \left(x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}} \right) \tag{16}$$
$$= 3 \tag{17}$$

Note that the total charge in the shell $q_s = 4\pi R^2 kR$ is equal and opposite to the total volume charge $\frac{4}{3}\pi R^3 (-3k)$.

Rather than working out the integrals above, we can use Gauss's law to find the electric field from a spherical shell and a uniformly charged sphere. We've already solved this problem in examples 1 and 2 here, so we can just quote the results. Inside the sphere, the spherical shell contributes nothing, and the volume charge gives a radial field of

$$E = -\frac{kr}{\epsilon_0} \tag{18}$$

Outside the sphere, the shell contributes

$$E_s = \frac{kR^3}{\epsilon_0 r^2} \tag{19}$$

and the volume charge contributes

$$E_v = -\frac{kR^3}{\epsilon_0 r^2} \tag{20}$$

so the total field is $E_s + E_v = 0$.

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