POWER RADIATED BY RADIATION REACTION FORCE

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The radiation reaction force is given, for a particle momentarily at rest, by the Abraham-Lorentz formula

$$\mathbf{F}_{rad}=\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}} \tag{1}$$

The more general formula for a charge moving with an arbitrary velocity is

$$F_{rad}=\frac{\mu_0 q^2\gamma^4}{6\pi c}\left(\dot{a}+\frac{3a^2\gamma^2 v}{c^2}\right) \tag{2}$$

The power radiated by a moving charge is given by

$$P=\frac{\mu_0 q^2\gamma^6}{6\pi c}\left[a^2-\frac{|\mathbf{v}\times\mathbf{a}|^2}{c^2}\right] \tag{3}$$

which reduces to the following formula if $\mathbf{v}$ and $\mathbf{a}$ are parallel:

$$P=\frac{\mu_0 q^2\gamma^6 a^2}{6\pi c} \tag{4}$$

The Abraham-Lorentz formula was derived by assuming that the charge is in the same state (effectively, that its velocity and acceleration are the same) at two points in time, and then calculating the average power emitted over that interval. That is, we assume

$$\int_{t_1}^{t_2}\mathbf{F}_{rad}\cdot\mathbf{v}dt=-\int_{t_1}^{t_2}Pdt \tag{5}$$

We can check that the general formula 2 is consistent with 4 under this assumption. Since the motion is in one dimension, we want to show that

$$\int_{t_1}^{t_2}\frac{\mu_0 q^2\gamma^4}{6\pi c}\left(\dot{a}+\frac{3a^2\gamma^2 v}{c^2}\right)v\,dt=-\int_{t_1}^{t_2}\frac{\mu_0 q^2\gamma^6 a^2}{6\pi c}dt \tag{6}$$

$$\int_{t_1}^{t_2}\left(\gamma^4\dot{a}v+\frac{3a^2\gamma^6 v^2}{c^2}\right)dt=-\int_{t_1}^{t_2}\gamma^6 a^2 dt \tag{7}$$

We'll need the following formula to do the integrals:

$$\dot{\gamma} = \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} \tag{8}$$

$$= \frac{v\dot{v}}{c^2\left(1-v^2/c^2\right)^{3/2}} \tag{9}$$

$$= \gamma^3\frac{va}{c^2} \tag{10}$$

Looking at the first term of the integral on the LHS, we can integrate by parts.

$$\int_{t_1}^{t_2}\gamma^4\dot{a}v\,dt = \gamma^4 av\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} a\left(a\gamma^4 + v\frac{d}{dt}\gamma^4\right)dt \tag{11}$$

$$= -\int_{t_1}^{t_2} a\left(a\gamma^4 + \frac{4av^2\gamma^6}{c^2}\right)dt \tag{12}$$

$$= -\int_{t_1}^{t_2} a^2\gamma^6\left(\gamma^{-2} + \frac{4v^2}{c^2}\right)dt \tag{13}$$

$$= -\int_{t_1}^{t_2} a^2\gamma^6\left(1 - \frac{v^2}{c^2} + \frac{4v^2}{c^2}\right)dt \tag{14}$$

$$= -\int_{t_1}^{t_2} a^2\gamma^6\left(1 + \frac{3v^2}{c^2}\right)dt \tag{15}$$

The integrated term in line 1 is zero because v and a are the same at both limits by assumption.

Adding this to the second term on the LHS of 7 we get

$$\int_{t_1}^{t_2}\left(\gamma^4\dot{a}v + \frac{3a^2\gamma^6v^2}{c^2}\right)dt = -\int_{t_1}^{t_2} a^2\gamma^6\left(1 + \frac{3v^2}{c^2} + \frac{3v^2}{c^2}\right)dt \tag{16}$$

$$= -\int_{t_1}^{t_2} a^2\gamma^6 dt \tag{17}$$

so equation 7 is confirmed.