POYNTING VECTOR

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If we want to calculate the work done by electromagnetic forces on a collection of charges and currents, we can start with the Lorentz force law

$$\mathbf{F}=q\mathbf{E}+q\mathbf{v}\times\mathbf{B} \tag{1}$$

In order for work to be done, the force has to act over a displacement of the charge, so in time dt the charge moves $\mathbf{v}dt$. As the magnetic force always acts perpendicular to the velocity, it does no work so all the work comes from the electric field:

$$dW=\mathbf{F}\cdot\mathbf{v}dt=q\mathbf{E}\cdot\mathbf{v}dt \tag{2}$$

If we now consider a continuous distribution of charge with density ρ then we can find the work done on a volume element $d^3\mathbf{r}$ by replacing q with $\rho d^3\mathbf{r}$. Also, the current density $\mathbf{J}$ is the charge density times the velocity, $\mathbf{J}=\rho\mathbf{v}$, so we get

$$dW = \left(\rho d^3\mathbf{r}\right)\mathbf{E}\cdot\mathbf{v}dt \tag{3}$$

$$= \mathbf{E}\cdot\mathbf{J}d^3\mathbf{r}dt \tag{4}$$

The rate at which work is done over a volume $\mathcal{V}$ is therefore

$$\frac{dW}{dt}=\int_{\mathcal{V}}\mathbf{E}\cdot\mathbf{J}d^3\mathbf{r} \tag{5}$$

We can write this entirely in terms of $\mathbf{E}$ and $\mathbf{B}$ by using one of Maxwell's equations

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{6}$$

Thus

$$\mathbf{E}\cdot\mathbf{J}=\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})-\epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} \tag{7}$$

An identity from vector calculus says

$$\nabla\cdot(\mathbf{E}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{B}) \tag{8}$$

so

$$\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})=\frac{1}{\mu_0}\left[\mathbf{B}\cdot(\nabla\times\mathbf{E})-\nabla\cdot(\mathbf{E}\times\mathbf{B})\right] \tag{9}$$

Using another of Maxwell's equations (Faraday's law)

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{10}$$

we get

$$\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})=-\frac{1}{\mu_0}\left[\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}+\nabla\cdot(\mathbf{E}\times\mathbf{B})\right] \tag{11}$$

For any vector field $\mathbf{A}$ we have

$$\frac{\partial}{\partial t}A^2=\frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{A})=2\mathbf{A}\cdot\frac{\partial}{\partial t}\mathbf{A} \tag{12}$$

so

$$\frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B})=-\frac{1}{\mu_0}\left[\frac{1}{2}\frac{\partial B^2}{\partial t}+\nabla\cdot(\mathbf{E}\times\mathbf{B})\right] \tag{13}$$

Combining this with 12 for $\mathbf{A}=\mathbf{E}$ and 7 we get

$$\mathbf{E}\cdot\mathbf{J}=-\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t}-\frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t}-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) \tag{14}$$

The total rate of work done in volume $\mathcal{V}$ is therefore

$$\frac{dW}{dt}=-\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}}\left(\frac{1}{\mu_0}B^2+\epsilon_0E^2\right)d^3\mathbf{r}-\frac{1}{\mu_0}\int_{\mathcal{V}}\nabla\cdot(\mathbf{E}\times\mathbf{B})\,d^3\mathbf{r} \tag{15}$$

Using the divergence theorem on the last integral, we get

$$\boxed{\frac{dW}{dt}=-\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}}\left(\frac{1}{\mu_0}B^2+\epsilon_0E^2\right)d^3\mathbf{r}-\frac{1}{\mu_0}\int_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}} \tag{16}$$

where the integral is now over the surface $\mathcal{S}$ enclosing the volume $\mathcal{V}$.

We've seen earlier that the energy stored in electric and magnetic fields is

$$W_E = \frac{\epsilon_0}{2}\int_{\mathcal{V}} E^2 d^3\mathbf{r} \tag{17}$$

$$W_B = \frac{1}{2\mu_0}\int_{\mathcal{V}} B^2 d^3\mathbf{r} \tag{18}$$

so the first integral in 16 is the negative of the rate of change of the energy stored in the two fields. That is, as the fields do work on the charges they expend their energy (so it decreases, giving a negative derivative) but the work done on the charges is positive. The second integral is the rate at which energy flows across the surface $\mathcal{S}$. This result is known as *Poynting's theorem*, named after John Henry Poynting (1852-1914), an English physicist.

The vector field

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}} \tag{19}$$

is called the *Poynting vector*, and represents the rate per unit area at which energy crosses a surface.

Example 1. We have a coaxial cable with inner radius a and outer radius b that carries a current I down the inner cylinder and $-I$ back on the outer cylinder. There is a potential difference V between the inner and outer cylinders.

This is a magnetostatic situation, so we can apply Ampère's law to find $\mathbf{B}$. The magnetic field is circumferential so we can choose a circular path of integration at a radius r between the two cylinders

$$2\pi rB = \mu_0 I \tag{20}$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}} \tag{21}$$

We can find the electric field from Gauss's law. If the linear charge density on the inner cylinder is ρ then by symmetry the electric field points radially outward from the axis of the cable. If we take a cylindrical Gaussian surface of radius r and length 1, then

$$2\pi rE = \frac{\rho}{\epsilon_0} \tag{22}$$

$$E = \frac{\rho}{2\pi\epsilon_0 r} \tag{23}$$

In order for the potential difference between the cylinders to be V, we must have

$$V = \int_a^b E dr \tag{24}$$

$$= \frac{\rho}{2\pi\epsilon_0}\int_a^b \frac{dr}{r} \tag{25}$$

$$= \frac{\rho}{2\pi\epsilon_0}\ln\frac{b}{a} \tag{26}$$

$$\rho = \frac{2\pi\epsilon_0 V}{\ln(b/a)} \tag{27}$$

$$\mathbf{E} = \frac{V}{r\ln(b/a)}\hat{\mathbf{r}} \tag{28}$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{29}$$

$$= \frac{IV}{2\pi r^2\ln(b/a)}\hat{\mathbf{z}} \tag{30}$$

The total rate at which power flows along the cable is then

$$\int \mathbf{S}\cdot d\mathbf{a} = \frac{2\pi IV}{2\pi\ln(b/a)}\int_a^b \frac{rdr}{r^2} \tag{31}$$

$$= IV \tag{32}$$

This is a special case of the general formula known as the *Joule heating law*, which is that the rate at which energy flows (the power) is $P = IV$.

Example 2. We have a transmission line consisting of two thin ribbons (each of width w) separated by an insulating vacuum space of thickness $h \ll w$, with each ribbon carrying current I (in opposite directions) and held at a potential difference V. Because the separation of the two ribbons is much less than their width, we can approximate the situation by considering each ribbon to be a sheet of current. This gives a magnetic field from each ribbon perpendicular to the cable. Take the cable to lie along the x axis, and the xy plane to be the plane containing the ribbons. Outside the ribbons there will be essentially zero magnetic field, since we can take an Ampèrian loop around both ribbons, and this loop will contain zero net current (since the currents in the ribbons flow in opposite directions). Between the ribbons, however, the magnetic field from one ribbon will be equal to that from the other ribbon, so the fields add up rather than cancel. We choose a rectangular path that encloses a cross-section of one of the ribbons. If we make the rectangle infinitesimally thin so that it two sides of length w and the other two sides of length $dz\approx0$, the path length is $2w$ and the enclosed current is I, so we have

$$\mathbf{B}_{r}=\frac{\mu_{0}I}{2w}\hat{\mathbf{y}} \tag{33}$$

The fields from the two ribbons will add in the space between them so the total field is

$$\mathbf{B}=2\mathbf{B}_{r}=\frac{\mu_{0}I}{w}\hat{\mathbf{y}} \tag{34}$$

The electric field is

$$\mathbf{E}=\frac{V}{h}\hat{\mathbf{z}} \tag{35}$$

so

$$\mathbf{S} = \frac{1}{\mu_{0}}\mathbf{E}\times\mathbf{B} \tag{36}$$

$$= -\frac{IV}{hw}\hat{\mathbf{x}} \tag{37}$$

and the total rate of energy flow is this times the area between the ribbons:

$$\int\mathbf{S}\cdot d\mathbf{a}=-\frac{IV}{hw}hw=-IV \tag{38}$$

The minus sign is there because of the direction we chose for the x axis; because the cable is symmetric, it just means that the power flows along the cable at a rate of IV, as we'd expect.

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