POYNTING'S THEOREM AND CONSERVATION OF ENERGY

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We've seen that Poynting's theorem allows us to calculate the rate at which energy changes within a given volume due to the action of electromagnetic fields. The theorem says

$$\frac{dW}{dt}=-\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{V}}\left(\frac{1}{\mu_0}B^2+\epsilon_0E^2\right)d^3\mathbf{r}-\frac{1}{\mu_0}\int_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \tag{1}$$

The first term gives the rate at which energy stored within the fields changes within the volume $\mathcal{V}$ and the second term gives the rate at which energy flows across the surface $\mathcal{S}$ enclosing $\mathcal{V}$.

As an example, we can revisit the following problem. We have a solid wire with a circular cross-section of radius a that has a narrow gap of width $w \ll a$ cut into it. If a steady current I is passed along the wire, the two surfaces on either side of the gap act as a parallel plate capacitor, as charge builds up on these surfaces. As a result, the electric field between the plates changes as the charge builds up, so we should get an induced magnetic field. We saw before that the electric field and its derivative within the gap are

$$\frac{\partial E}{\partial t}=\frac{I}{\pi a^2\epsilon_0} \tag{2}$$

$$\mathbf{E}(r,t)=\frac{I}{\pi a^2\epsilon_0}t\hat{\mathbf{z}} \tag{3}$$

assuming that $E(0)=0$, and the z axis is parallel to the wire, pointing from the positive side of the gap to the negative. The electric field is independent of the distance r from the axis.

The magnetic field is given by

$$\mathbf{B}(r,t)=\mu_0\frac{Ir}{2\pi a^2}\hat{\boldsymbol{\phi}} \tag{4}$$

Because $\frac{\partial E}{\partial t}$ is independent of time, so is $\mathbf{B}$, although it does depend on r.

The energy density u_{em} in the fields is then

$$u_{\text{em}} = \frac{1}{2}\left(\frac{1}{\mu_0}B^2 + \epsilon_0 E^2\right) \tag{5}$$
$$= \frac{1}{2}\left(\frac{\mu_0 I^2 r^2}{4\pi^2 a^4} + \frac{I^2 t^2}{\pi^2 a^4 \epsilon_0}\right) \tag{6}$$
$$= \frac{I^2}{2\pi^2 a^4}\left(\frac{\mu_0 r^2}{4} + \frac{t^2}{\epsilon_0}\right) \tag{7}$$

Because of the increasing magnetic field, the energy density increases with time.

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{8}$$
$$= -\frac{I^2 r t}{2\pi^2 a^4 \epsilon_0}\hat{\mathbf{r}} \tag{9}$$

Note that **S** points inwards, indicating that energy is flowing into the gap in the wire, causing the increase in the energy density.

In general, if there is any charge in the volume $\mathcal{V}$, then the electric field will do work on the charge if the charge moves around (magnetic fields do no work). This work will change the mechanical energy u_{mech} of the charges (that is, their potential or kinetic energy). The total energy flowing into the volume must therefore add up to the sum of u_{mech} and u_{em}; that is, the energy flowing into a volume is stored either in the energy of the fields or in the mechanical energy of the charge. Since the Poynting vector **S** gives the *rate* at which energy flows across a surface, the integral of **S** over the surface bounding $\mathcal{V}$ must equal the rate of change of $u_{\text{mech}} + u_{\text{em}}$, that is

$$-\oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a} = \frac{d}{dt}\int_{\mathcal{V}} (u_{\text{mech}} + u_{\text{em}})\, d^3\mathbf{r} \tag{10}$$

where the minus sign on the LHS is there because we want the energy flowing into the volume and $d\mathbf{a}$ is an outward pointing normal to the surface. Using the divergence theorem to convert the LHS to a volume integral, we get the differential form:

$$-\nabla\cdot\mathbf{S} = \frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) \tag{11}$$

In our present example, there is no charge in the gap, so $u_{\text{mech}} = 0$ so we should have

$$-\nabla\cdot\mathbf{S}=\frac{\partial u_{\text{em}}}{\partial t} \tag{12}$$

We can check this:

$$-\nabla\cdot\mathbf{S}=-\frac{1}{r}\frac{\partial}{\partial r}\left(rS_r\right) \tag{13}$$

$$=\frac{1}{r}\frac{2I^2rt}{2\pi^2a^4\epsilon_0} \tag{14}$$

$$=\frac{I^2t}{\pi^2a^4\epsilon_0} \tag{15}$$

$$=\frac{\partial u_{\text{em}}}{\partial t} \tag{16}$$

so it checks out.

The total energy in the gap as a function of time can be found by integrating 7 over the volume of the gap:

$$u_{\text{em}}(t)=\frac{I^2}{2\pi^2a^4}\int_{\mathcal{V}}\left(\frac{\mu_0r^2}{4}+\frac{t^2}{\epsilon_0}\right)d^3\mathbf{r} \tag{17}$$

$$=\frac{2\pi w\mu_0I^2}{8\pi^2a^4}\int_0^a r^3dr+\frac{\pi a^2wI^2t^2}{2\pi^2a^4\epsilon_0} \tag{18}$$

$$=\frac{\mu_0wI^2}{16\pi}+\frac{wI^2t^2}{2\pi a^2\epsilon_0} \tag{19}$$

$$\frac{\partial u_{\text{em}}}{\partial t}=\frac{wI^2t}{\pi a^2\epsilon_0} \tag{20}$$

$$-\oint_{\mathcal{S}}\mathbf{S}\cdot d\mathbf{a}=\frac{I^2at}{2\pi^2a^4\epsilon_0}\left(2\pi aw\right) \tag{21}$$

$$=\frac{wI^2t}{\pi a^2\epsilon_0} \tag{22}$$

$$=\frac{\partial u_{\text{em}}}{\partial t} \tag{23}$$

Thus again it checks out.