QUADRUPOLE MOMENT

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We've seen that the dipole term in the multipole expansion can be written as

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \tag{1}$$

$$= \frac{1}{4\pi\epsilon_0 r^2}\sum_{i=1}^{3}\hat{r}_i p_i \tag{2}$$

where $\mathbf{p}$ is the dipole moment and the subscript i indicates the ith component of the corresponding vector. In particular

$$p_i = \int r_i'\rho(\mathbf{r}')d^3\mathbf{r}' \tag{3}$$

We can use a similar derivation to define a *quadrupole moment*. The quadrupole term in the multipole expansion is

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r'^2 P_2(\cos\theta')\rho(\mathbf{r}')d^3\mathbf{r}' \tag{4}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{5}$$

The angle θ' is the angle between the vectors $\mathbf{r}'$ and $\mathbf{r}$, so $r'\cos\theta' = \mathbf{r}'\cdot\hat{\mathbf{r}}$ where the 'hat' denotes a unit vector. We can therefore write the quadrupole term as

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(3\left(\mathbf{r}'\cdot\hat{\mathbf{r}}\right)^2 - \mathbf{r}'\cdot\mathbf{r}'\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{6}$$

In terms of components

$$\mathbf{r}'\cdot\hat{\mathbf{r}} = \sum_{i=1}^{3} r_i'\hat{r}_i \tag{7}$$

$$\mathbf{r}'\cdot\mathbf{r}' = r'^2 \tag{8}$$

so we can write

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\int\left(3\left(\sum_{i=1}^{3} r'_i\hat{r}_i\right)\left(\sum_{j=1}^{3} r'_j\hat{r}_j\right) - r'^2\right)\rho(\mathbf{r}')d^3\mathbf{r}' \quad (9)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\int\left(3r'_ir'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}')d^3\mathbf{r}' \quad (10)$$

$$\equiv \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_jQ_{ij} \quad (11)$$

where in the second line δ_{ij} is the usual Kronecker delta symbol and we used the fact that

$$\sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\delta_{ij} = 1 \quad (12)$$

since $\hat{\mathbf{r}}$ is a unit vector.

The quadrupole moment is defined by

$$Q_{ij} \equiv \int\left(3r'_ir'_j - \left(r'_i\right)^2\delta_{ij}\right)\rho(\mathbf{r}')d^3\mathbf{r}' \quad (13)$$

The quadrupole moment is a second-rank tensor, which can be represented by a 3×3 symmetric matrix.

Example 1. As an example, consider a configuration of four point charges at the corners of a square of side a in the xy plane, with $+q$ at locations $(a/2, a/2)$ and $(-a/2, -a/2)$, and $-q$ at $(-a/2, a/2)$ and $(a/2, -a/2)$. Then we have

$$Q_{11} = q\frac{a^2}{4}(-2+2-2+2) \quad (14)$$

$$= 0 \quad (15)$$

We can work out the other components in the same way and get

$$Q = q\frac{a^2}{4}\begin{bmatrix} 0 & 12 & 0 \\ 12 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3qa^2 & 0 \\ 3qa^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (16)$$

How does the quadrupole moment depend on the choice of origin? Suppose we shift the origin by an amount $\mathbf{a}$. Then

$$Q'_{ij} = \int \left(3\left(r'_i + a_i\right)\left(r'_j + a_j\right) - \left(r'_i + a_i\right)^2 \delta_{ij}\right) \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' \tag{17}$$

$$= \int \left(3r'_i r'_j - \left(r'_i\right)^2 \delta_{ij}\right) \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' + \tag{18}$$

$$3a_i \int r'_j \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' + 3a_j \int r'_i \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' - \tag{19}$$

$$2a_i\delta_{ij} \int r'_i \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' + \tag{20}$$

$$3a_i a_j \int \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' - a_i^2 \delta_{ij} \int \rho(\mathbf{r}' + \mathbf{a}) d^3\mathbf{r}' \tag{21}$$

The first term in the expanded integral (line 2) is actually just Q_{ij}, since we are multiplying the charge at location $\mathbf{r}' + \mathbf{a}$ (which in the old coordinates was at $\mathbf{r}'$) by the old coordinates. By the same argument, the three terms in lines 3 and 4 are all constants multiplied by the dipole moment, and the two terms in line 5 are constants multiplied by the monopole moment (which is just the total charge). Thus if both the dipole and monopole moments are zero, then the quadrupole moment is independent of the origin.

We can use the same approach to define higher order moments. For example, the next moment in the series is the octopole moment. To define it, we start with the octopole term in the expansion:

$$V_{\text{oct}} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^4} \int r'^3 P_3(\cos\theta') \rho(\mathbf{r}') d^3\mathbf{r}' \tag{22}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{2r^4} \int r'^3 \left(5\cos^3\theta' - 3\cos\theta'\right) \rho(\mathbf{r}') d^3\mathbf{r}' \tag{23}$$

For the first term, we get

$$5r'^3 \cos^3\theta' = 5\left(\mathbf{r}' \cdot \hat{\mathbf{r}}\right)^3 \tag{24}$$

$$= 5 \sum_{i,j,k=1}^{3} \hat{r}_i \hat{r}_j \hat{r}_k r'_i r'_j r'_k \tag{25}$$

For the second term, we have

$$3r'^3\cos\theta' = 3r'^2 r'\cos\theta' \tag{26}$$
$$= 3r'^2\mathbf{r}'\cdot\hat{\mathbf{r}} \tag{27}$$
$$= 3r'^2\sum_{i=1}^{3} r'_i\hat{r}_i \tag{28}$$
$$= 3r'^2\sum_{i,j,k=1}^{3}\hat{r}_i\hat{r}_j\hat{r}_k\delta_{jk}r'_i \tag{29}$$

where in the last line we used 12 above.

Putting these two expansions back into the expression for V_{oct} we get

$$V_{\text{oct}} = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^4}\sum_{i,j,k=1}^{3}\hat{r}_i\hat{r}_j\hat{r}_k\int\left(5r'_ir'_jr'_k - 3r'^2r'_i\delta_{jk}\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{30}$$

We can therefore define an octopole moment as the integral term:

$$O_{ijk} = \int\left(5r'_ir'_jr'_k - 3r'^2r'_i\delta_{jk}\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{31}$$

Note that this isn't the only way we could have defined it, since the second term above could be written in various ways. It is probably more traditional to write it so that it is symmetric with respect to all three of its indexes. That is, we could write it as

$$3r'^3\cos\theta' = r'^2\sum_{i,j,k=1}^{3}\hat{r}_i\hat{r}_j\hat{r}_k\left[\delta_{jk}r'_i + \delta_{ik}r'_j + \delta_{ij}r'_k\right] \tag{32}$$

which makes the octopole moment come out to

$$O_{ijk} = \int\left(5r'_ir'_jr'_k - r'^2\left[\delta_{jk}r'_i + \delta_{ik}r'_j + \delta_{ij}r'_k\right]\right)\rho(\mathbf{r}')d^3\mathbf{r}' \tag{33}$$

The two forms do have different components, but they are equally valid since when placed into the original sum they both give the same potential.