RADIATION FROM A CURRENT LOOP WITH TIME-VARYING CURRENT

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We've looked at the fields produced by a magnetic dipole that oscillates with a regular frequency ω. By following the procedure in Griffiths's section 11.1.4, where he derives the fields due to an electric dipole of arbitrary shape, we can derive the formulas for a magnetic dipole consisting of a circular current loop carrying a time-dependent current $I(t)$ where the time dependence is arbitrary.

We assume the current loop has radius b and lies in the xy plane with its centre on the z axis. Since at any instant, the magnitude of the current is the same everywhere in the loop, we can use the same argument as in the oscillating case to deduce that for some observation point **r** in the xz plane, the vector potential **A** points in the y direction, and thus, since **A** is always tangential to the loop, its direction in general is in the ϕ direction. If the loop is electrically neutral, the electric potential is $V = 0$, so we need to calculate only **A**. The retarded potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{I(t-d/c)}{d}d\boldsymbol{\ell}' \tag{1}$$

$$= \frac{\mu_0 b}{4\pi}\hat{\boldsymbol{\phi}}\int_0^{2\pi}\frac{I(t-d/c)}{d}\cos\phi' d\phi' \tag{2}$$

where ϕ' is the azimuthal angle around the loop so that the y component of $d\boldsymbol{\ell}'$ is $b\cos\phi'$ and the retarded time is

$$t_r \equiv t - \frac{d}{c} \tag{3}$$

and

$$d \equiv \left|\mathbf{r}-\mathbf{r}'\right| \tag{4}$$

$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \tag{5}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{r}-\mathbf{r}'}{d} \tag{6}$$

where $\mathbf{r}'$ is the position on the loop being integrated over.

For our observation point in the xz plane, we have

$$\mathbf{r} = r\sin\theta\hat{\mathbf{x}} + r\cos\theta\hat{\mathbf{z}} \tag{7}$$

and for a point on the loop

$$\mathbf{r}' = b\cos\phi'\hat{\mathbf{x}} + b\sin\phi'\hat{\mathbf{y}} \tag{8}$$

In what follows, we'll use the notation $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$, etc to simplify the notation.

Therefore, assuming $b \ll r$ (the loop is very small)

$$d = \sqrt{r^2 + b^2 - 2\mathbf{r}\cdot\mathbf{r}'} \tag{9}$$

$$\cong r\left(1 - \frac{b}{r}s_\theta c_{\phi'}\right) \tag{10}$$

$$\frac{1}{d} \cong \frac{1}{r}\left(1 + \frac{b}{r}s_\theta c_{\phi'}\right) \tag{11}$$

We can expand the current in a Taylor series about $t_0 \equiv t - \frac{r}{c}$:

$$I\left(t - \frac{d}{c}\right) \cong I\left(t - \frac{r}{c} + \frac{b}{c}s_\theta c_{\phi'}\right) \tag{12}$$

$$= I(t_0) + \dot{I}(t_0)\frac{b}{c}s_\theta c_{\phi'} + \frac{1}{2!}\ddot{I}(t_0)\left(\frac{b}{c}s_\theta c_{\phi'}\right)^2 + \ldots \tag{13}$$

We are justified in dropping the last term if

$$\frac{1}{2!}\ddot{I}(t_0)\left(\frac{b}{c}s_\theta c_{\phi'}\right)^2 \ll \dot{I}(t_0)\frac{b}{c}s_\theta c_{\phi'} \tag{14}$$

$$b \ll \frac{c}{\left|\ddot{I}/\dot{I}\right|} \tag{15}$$

If we compare higher derivative terms with the first order term, we get the general condition

$$b \ll c\left|\frac{\dot{I}}{d^nI/dt^n}\right|^{n-1} \tag{16}$$

Assuming this is true, we get from 2:

$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0 b}{4\pi r}\hat{\boldsymbol{\phi}}\int_0^{2\pi}\left(I(t_0)+\dot{I}(t_0)\frac{b}{c}s_\theta c_{\phi'}\right)\left(1+\frac{b}{r}s_\theta c_{\phi'}\right)c_{\phi'}d\phi' \quad (17)$$

$$\cong \frac{\mu_0\pi b^2}{4\pi r}\left(\frac{I(t_0)}{r}+\frac{\dot{I}(t_0)}{c}\right)s_\theta\hat{\boldsymbol{\phi}} \quad (18)$$

where to get the second line, we discarded the term in b^3 and used $\int_0^{2\pi}\cos\phi' d\phi' = 0$ and $\int_0^{2\pi}\cos^2\phi' d\phi' = \pi$. If we're interested only in the radiation produced by this dipole, we can ignore any terms in the potential that are of order 2 or higher in $\frac{1}{r}$, since it is only $\frac{1}{r^2}$ terms in the Poynting vector that will contribute to radiation that escapes to infinity. Therefore, we can throw away the first term above to get our final approximation:

$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0\pi b^2}{4\pi rc}\dot{I}(t_0)s_\theta\hat{\boldsymbol{\phi}} \quad (19)$$

We can write this in terms of the magnetic moment of the loop, which is

$$m(t_0) = \pi b^2 I(t_0) \quad (20)$$

so

$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0}{4\pi rc}\dot{m}(t_0)s_\theta\hat{\boldsymbol{\phi}}$$

We can now calculate the fields:

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} \quad (21)$$

$$= -\frac{\mu_0 s_\theta}{4\pi rc}\ddot{m}(t_0)\hat{\boldsymbol{\phi}} \quad (22)$$

$$\mathbf{B} = \nabla\times\mathbf{A} \quad (23)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}(rA)\hat{\boldsymbol{\theta}} \quad (24)$$

$$= -\frac{\mu_0 s_\theta}{4\pi rc}\frac{\partial\dot{m}}{\partial r}\hat{\boldsymbol{\theta}} \quad (25)$$

where in calculating $\mathbf{B}$, we ignored the term $\frac{1}{rs_\theta}\frac{\partial}{\partial\theta}(s_\theta A)\hat{\mathbf{r}}$ since it gives a term containing $\frac{1}{r^2}$.

Since $m = m\left(t-\frac{r}{c}\right)$ we have

$$\frac{\partial\dot{m}}{\partial r} = -\frac{1}{c}\ddot{m} \quad (26)$$

$$\mathbf{B} = \frac{\mu_0 s_\theta}{4\pi rc^2}\ddot{m}\hat{\boldsymbol{\theta}} \quad (27)$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{28}$$

$$= \frac{\mu_0 s_\theta^2 \ddot{m}^2}{16\pi^2 r^2 c^3}\hat{\mathbf{r}} \tag{29}$$

The power radiated is the integral of **S** over a large sphere of radius r:

$$P = \int \mathbf{S}\cdot d\mathbf{a} \tag{30}$$

$$= \frac{2\pi\mu_0 \ddot{m}^2}{16\pi^2 c^3}\int_0^\pi \frac{\sin^2\theta}{r^2} r^2 \sin\theta d\theta \tag{31}$$

$$= \frac{\mu_0 \ddot{m}^2}{6\pi c^3} \tag{32}$$