

## RADIATION FROM A ROTATING DIPOLE

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 13 August 2021.

We can simulate a rotating dipole by superimposing two perpendicular oscillating dipoles. If our rotating dipole is located at the origin and rotates about the  $z$  axis (so the axis of the dipole lies in the  $xy$  plane), then we get

$$\mathbf{p} = p_0 (\cos \omega t \hat{\mathbf{x}} + \sin \omega t \hat{\mathbf{y}}) \quad (1)$$

Since the fields obey the superposition principle (fields from 2 sources just add), we can work through the formulas we found earlier to get the fields and thus the radiated power. To simplify the notation, we'll use the shorthand

$$c_\omega \equiv \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \quad (2)$$

$$c_\theta \equiv \cos \theta \quad (3)$$

$$c_\phi \equiv \cos \phi \quad (4)$$

with analogous notation for the sines of these quantities.

The fields for a dipole pointing in an arbitrary direction are

$$\mathbf{E} = -\frac{\mu_0 \omega^2}{4\pi r} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) (\hat{\mathbf{p}}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} \quad (5)$$

$$\mathbf{B} = -\frac{\mu_0 \omega^2}{4\pi r c} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) (\hat{\mathbf{p}}_0 \times \hat{\mathbf{r}}) \quad (6)$$

Superposing the two perpendicular dipoles we get

$$\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} [(c_\omega \hat{\mathbf{x}} + s_\omega \hat{\mathbf{y}}) \times \hat{\mathbf{r}}] \times \hat{\mathbf{r}} \quad (7)$$

$$\mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} [(c_\omega \hat{\mathbf{x}} + s_\omega \hat{\mathbf{y}}) \times \hat{\mathbf{r}}] \quad (8)$$

To do the cross products we convert the rectangular unit vectors to spherical unit vectors:

$$\hat{\mathbf{x}} = s_\theta c_\phi \hat{\mathbf{r}} + c_\theta c_\phi \hat{\boldsymbol{\theta}} - s_\phi \hat{\boldsymbol{\phi}} \quad (9)$$

$$\hat{\mathbf{y}} = s_\theta s_\phi \hat{\mathbf{r}} + c_\theta s_\phi \hat{\boldsymbol{\theta}} + c_\phi \hat{\boldsymbol{\phi}} \quad (10)$$

Then

$$\hat{\mathbf{x}} \times \hat{\mathbf{r}} = -c_\theta c_\phi \hat{\boldsymbol{\phi}} - s_\phi \hat{\boldsymbol{\theta}} \quad (11)$$

$$(\hat{\mathbf{x}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -c_\theta c_\phi \hat{\boldsymbol{\theta}} + s_\phi \hat{\boldsymbol{\phi}} \quad (12)$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{r}} = -c_\theta s_\phi \hat{\boldsymbol{\phi}} + c_\phi \hat{\boldsymbol{\theta}} \quad (13)$$

$$(\hat{\mathbf{y}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -c_\theta s_\phi \hat{\boldsymbol{\theta}} - c_\phi \hat{\boldsymbol{\phi}} \quad (14)$$

Plugging everything in and collecting terms we get

$$\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \left[ \hat{\boldsymbol{\theta}} c_\theta (-c_\omega c_\phi - s_\omega s_\phi) + \hat{\boldsymbol{\phi}} (c_\omega s_\phi - s_\omega c_\phi) \right] \quad (15)$$

$$\mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} \left[ -\hat{\boldsymbol{\theta}} (c_\omega s_\phi - s_\omega c_\phi) + \hat{\boldsymbol{\phi}} c_\theta (-c_\omega c_\phi - s_\omega s_\phi) \right] \quad (16)$$

Defining

$$E_\theta \equiv -c_\theta (c_\omega c_\phi + s_\omega s_\phi) \quad (17)$$

$$E_\phi \equiv (c_\omega s_\phi - s_\omega c_\phi) \quad (18)$$

we can write the fields as

$$\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \left[ \hat{\boldsymbol{\theta}} E_\theta + \hat{\boldsymbol{\phi}} E_\phi \right] \quad (19)$$

$$\mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} \left[ -\hat{\boldsymbol{\theta}} E_\phi + \hat{\boldsymbol{\phi}} E_\theta \right] \quad (20)$$

In this form, it's obvious that  $\mathbf{E} \cdot \mathbf{B} = 0$ , so  $\mathbf{E}$  and  $\mathbf{B}$  are perpendicular.

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad (21)$$

$$= \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 (E_\theta^2 + E_\phi^2) \hat{\mathbf{r}} \quad (22)$$

$$= \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 \left( c_\theta^2 (c_\omega c_\phi + s_\omega s_\phi)^2 + (c_\omega s_\phi - s_\omega c_\phi)^2 \right) \hat{\mathbf{r}} \quad (23)$$

The average energy radiated is the average of  $\mathbf{S}$  over a single time cycle, so it's the average over the terms involving  $c_\omega$  and  $s_\omega$ . These are of two types: terms involving  $c_\omega^2$  or  $s_\omega^2$  and the cross terms involving  $s_\omega c_\omega$ . The average of  $s_\omega c_\omega$  over a cycle is zero and the average of  $c_\omega^2$  or  $s_\omega^2$  is  $\frac{1}{2}$ , so the cross terms contribute nothing and we get

$$\langle \mathbf{S} \rangle = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 \left[ \frac{1}{2} c_\theta^2 (c_\phi^2 + s_\phi^2) + \frac{1}{2} (c_\phi^2 + s_\phi^2) \right] \quad (24)$$

$$= \frac{\mu_0}{2c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 (1 + \cos^2 \theta) \quad (25)$$

The average radiated power is maximum in the  $\pm z$  directions where  $\theta = 0, \pi$  and minimum (though not zero) in the  $xy$  plane, where  $\theta = \frac{\pi}{2}$ . There is no dependence on  $\phi$  which is what we'd expect on average since the dipole rotates uniformly through all values of  $\phi$ . [There *is* a dependence on  $\phi$  within each cycle, since the radiated power in a given azimuthal direction depends on where the dipole is in its rotation.]

The total average radiated power is

$$\langle P \rangle = \frac{\mu_0}{2c} \left( \frac{p_0 \omega^2}{4\pi} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{1 + \cos^2 \theta}{r^2} r^2 \sin \theta d\phi d\theta \quad (26)$$

$$= \frac{\mu_0 p_0^2 \omega^4}{6\pi c} \quad (27)$$

This is exactly twice the power from a single oscillating dipole. Although power doesn't ordinarily obey the superposition principle since it depends on the product of  $\mathbf{E}$  and  $\mathbf{B}$ , it does here because the cross terms in 23 average out to zero over a time cycle, since the two perpendicular dipoles are  $\frac{\pi}{2}$  out of phase. If they were exactly in phase, we would replace  $s_\omega$  by  $c_\omega$  everywhere in the calculation, and then the cross terms wouldn't average out to zero and the combined power would not be twice the individual power.

#### PINGBACKS

Pingback: Electric dipole radiation from an arbitrary source