RADIATION REACTION WITH A DELTA-FUNCTION EXTERNAL FORCE

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Here's another example of applying an external force to a charge feeling the radiation reaction force. In general, a charge's acceleration obeys the differential equation

$$a=\tau\dot{a}+\frac{F}{m} \tag{1}$$

where F is the external force and

$$\tau\equiv\frac{\mu_{0}q^{2}}{6\pi mc} \tag{2}$$

Suppose now that the force is a delta function:

$$F=k\delta\left(t\right) \tag{3}$$

for some constant k. In the earlier post, we showed that if F is finite everywhere, then a must be continuous everywhere. However, here F is not finite at $t=0$. As before we start by integrating 1 over a small time interval around $t=0$:

$$\int_{-\epsilon}^{\epsilon}a\,dt=\tau\left[a\left(\epsilon\right)-a\left(-\epsilon\right)\right]+\frac{1}{m}\int_{-\epsilon}^{\epsilon}F\,dt \tag{4}$$

Provided that a is finite everywhere, the integral on the LHS goes to zero as $\epsilon\to0$ so we're left with

$$\tau\Delta a=-\frac{1}{m}\int_{-\epsilon}^{\epsilon}F\,dt \tag{5}$$

$$=-\frac{k}{m}\int_{-\epsilon}^{\epsilon}\delta\left(t\right)\,dt \tag{6}$$

$$=-\frac{k}{m} \tag{7}$$

$$\Delta a=-\frac{k}{m\tau} \tag{8}$$

We can repeat the calculations we did earlier to check that energy is conserved here. Since $F=0$ everywhere except $t=0$, the general solution of 1 is

$$a(t)=\begin{cases}a_0e^{t/\tau} & t<0\\ a_1e^{t/\tau} & t>0\end{cases}\tag{9}$$

If we eliminate the runaway acceleration for $t>0$ by requiring $a_1=0$ then the condition 8 requires $a_0=k/m\tau$, so

$$a(t)=\begin{cases}\frac{k}{m\tau}e^{t/\tau} & t<0\\ 0 & t>0\end{cases}\tag{10}$$

By requiring $v=0$ at $t=-\infty$ and that v is continuous at $t=0$ we get

$$v(t)=\begin{cases}\frac{k}{m}e^{t/\tau} & t<0\\ \frac{k}{m} & t>0\end{cases}\tag{11}$$

The work done by the force is

$$W=\int_{-\infty}^{\infty}Fv\,dt\tag{12}$$

$$=k\int_{-\infty}^{\infty}\delta(t)\,v\,dt\tag{13}$$

$$=kv(0)\tag{14}$$

$$=\frac{k^2}{m}\tag{15}$$

The energy radiated R is given by integrating the Larmor formula

$$P=\frac{\mu_0q^2a^2}{6\pi c}=m\tau a^2\tag{16}$$

so we get

$$R=m\tau\int_{-\infty}^{0}\left(\frac{k}{m\tau}\right)^2e^{2t/\tau}dt\tag{17}$$

$$=\frac{k^2}{2m}\tag{18}$$

The final kinetic energy is

$$K=\frac{1}{2}m\frac{k^2}{m^2}=\frac{k^2}{2m}\tag{19}$$

Thus

$$W = R + K \tag{20}$$

and energy is conserved.