RADIATIVE DECAY OF THE BOHR HYDROGEN ATOM

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The instantaneous power radiated by an accelerating point charge is given by the Larmor formula (valid for a charge moving at a speed $v \ll c$):

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{1}$$

One historic application of this formula was to Bohr's early model of the hydrogen atom as an electron in a classical circular orbit around the proton, with the centripetal force provided by the Coulomb attraction. Since a particle moving in a circle is accelerating, it will radiate away energy, so its orbit should eventually decay until the electron crashes into the proton. This classical instability of atoms was one motivation behind the introduction of quantum theory, but that's another story. Here, we'll investigate how long it would take a Bohr hydrogen atom to decay.

First, we need to reassure ourselves that the electron is moving non-relativistically. From equating centripetal and Coulomb forces, we have

$$\frac{mv^2}{r} = \frac{q^2}{4\pi\epsilon_0 r^2} \tag{2}$$

$$v = \frac{q}{\sqrt{4\pi\epsilon_0 mr}} \tag{3}$$

$$= qc\sqrt{\frac{\mu_0}{4\pi mr}} \tag{4}$$

using $c = 1/\sqrt{\mu_0\epsilon_0}$. Plugging in the numbers we get

$$\frac{v}{c} = \frac{5.3\times 10^{-8}}{\sqrt{r}} \tag{5}$$

The Bohr radius is $r = 5\times 10^{-11}$ m so at that radius $v/c = 0.0075$ so we're safe here. As r gets smaller, of course, v will increase but since the dependence is on the square root, the rate of increase of v is fairly small, so that even when $r = a/100$, v has increased only to $0.075c$. So for most of its journey towards the proton, the electron is moving non-relativistically.

To work out how long it takes for the decay to occur, consider the energy radiated during a time dt, which is $P\,dt$. From conservation of energy, this must be equal to the amount of energy lost by the electron. The total energy of the electron is its kinetic plus potential energy, so

$$E = \frac{1}{2}mv^2 - \frac{q^2}{4\pi\epsilon_0 r} \quad (6)$$

$$= \frac{q^2c^2\mu_0}{8\pi r} - \frac{q^2c^2\mu_0}{4\pi r} \quad (7)$$

$$= -\frac{q^2c^2\mu_0}{8\pi r} \quad (8)$$

Therefore, the energy lost is

$$dE = -\frac{q^2c^2\mu_0}{8\pi r^2}dr \quad (9)$$

[We've taken dE as negative, since the electron loses energy.] Putting these results together, we get

$$Pdt = \frac{\mu_0 q^2 a^2}{6\pi c}dt \quad (10)$$

$$= -\frac{q^2c^2\mu_0}{8\pi r^2}dr \quad (11)$$

$$dt = -\frac{3c^3}{4r^2a^2}dr \quad (12)$$

We need a to solve this, but this is just the centripetal acceleration, so

$$a = \frac{v^2}{r} \quad (13)$$

$$= \frac{q^2c^2\mu_0}{4\pi m r^2} \quad (14)$$

Therefore,

$$dt = -\frac{12\pi^2 m^2}{q^4 c \mu_0^2} r^2 dr \tag{15}$$

$$= -3.159 \times 10^{20} r^2 dr \tag{16}$$

$$\int_0^T dt = -3.159 \times 10^{20} \int_a^0 r^2 dr \tag{17}$$

$$= \frac{1}{3}\left(3.159 \times 10^{20}\right)\left(5 \times 10^{-11}\right)^3 \tag{18}$$

$$T = 1.31 \times 10^{-11} \text{ s} \tag{19}$$

Thus in classical electrodynamics, the hydrogen atom is so unstable that it would decay in a tiny fraction of a second. We should be grateful that quantum mechanics saves the universe.