RETARDED POTENTIAL OF A WIRE LOOP

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One more example of calculating the retarded potential. We have a loop of wire in the following shape. It extends along the x axis from $-b$ to $-a$, then in a semicircular loop of radius a clockwise around to $x=+a$, then along the x axis from $+a$ to $+b$, then in a semicircular loop of radius b back to $x=-b$. A linearly increasing current

$$I(t)=kt \tag{1}$$

flows through the loop in the direction given above. Assuming the wire is electrically neutral, $V=0$ so our job is to find $\mathbf{A}$.

Calculating $\mathbf{A}$ in general is a complex task, so we'll look only at the value of $\mathbf{A}$ at the origin. Consider first the inner loop of radius a. All points on this loop are at the same distance a from the origin, so the retarded time is the same for all points on the loop. Since the current goes left to right around the semicircle, the contribution to $\mathbf{A}$ is

$$\mathbf{A}_a = \frac{\mu_0}{4\pi}\int_\pi^0 \frac{k\left(t-\frac{a}{c}\right)}{a}\hat{\boldsymbol{\theta}}a\,d\theta \tag{2}$$

$$= -\frac{\mu_0}{4\pi}k\left(t-\frac{a}{c}\right)\int_0^\pi\left(-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}\right)d\theta \tag{3}$$

$$= \frac{\mu_0}{2\pi}k\left(t-\frac{a}{c}\right)\hat{\mathbf{x}} \tag{4}$$

We get a similar expression for the loop around the outer semicircle except this time the current flows right to left so the sign is reversed:

$$\mathbf{A}_b = -\frac{\mu_0}{2\pi}k\left(t-\frac{b}{c}\right)\hat{\mathbf{x}} \tag{5}$$

Adding these two together we get

$$\mathbf{A}_{ab} = \frac{\mu_0}{2\pi}k\frac{b-a}{c}\hat{\mathbf{x}} \tag{6}$$

The contributions from each of the two horizontal segments are equal, so for these two segments we have

$$\mathbf{A}_x = 2\frac{\mu_0}{4\pi}k\hat{\mathbf{x}}\int_a^b \frac{t-\frac{x}{c}}{x}dx \tag{7}$$

$$= \frac{\mu_0}{2\pi}k\hat{\mathbf{x}}\left[t\ln\frac{b}{a}-\frac{b-a}{c}\right] \tag{8}$$

The total potential is then

$$\mathbf{A}(0,t) = \mathbf{A}_{ab}+\mathbf{A}_x \tag{9}$$

$$= \frac{\mu_0}{2\pi}k\hat{\mathbf{x}}t\ln\frac{b}{a} \tag{10}$$

Because we have the potential at only a single point in space, we can't calculate any of its derivatives, so we can't calculate $\mathbf{B}=\nabla\times\mathbf{A}$. However we can calculate $\mathbf{E}$:

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} \tag{11}$$

$$= -\frac{\mu_0}{2\pi}k\hat{\mathbf{x}}\ln\frac{b}{a} \tag{12}$$

The electric field is constant in time at the origin. An electrically neutral wire can produce an electric field since the changing current induces a changing magnetic field which in turn produces an electric field.