SOLENOID WITH ARBITRARY CROSS-SECTION

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We calculated the magnetic field of a solenoid with a circular cross-section and found that, for an infinite solenoid

$$B=\mu_0 nI \tag{1}$$

where n is the number of turns per unit length and I is the current flowing through the wire wrapped around the cylinder.

We can actually find the magnetic field of an infinite solenoid of any cross-section, provided that cross-section is constant over the entire length. First, we use the Biot-Savart law to find the direction of the field. We'll choose a rectangular coordinate system such that the axis of the solenoid is the y axis. This means that any cross-section of the solenoid is parallel to the xz plane, so the current has no y component (well, OK, it does have a very slight y component since the wire is wound as a helix and current does flow down the length of the solenoid, but we can assume here that we've got an idealized solenoid where the current flows in closed loops around the axis). That is

$$\mathbf{I}=I_x\hat{\mathbf{x}}+I_z\hat{\mathbf{z}} \tag{2}$$

Now pick a point on the solenoid at position $\mathbf{r}_1'=x'\hat{\mathbf{x}}+y'\hat{\mathbf{y}}+z'\hat{\mathbf{z}}$. If we put the observation point $\mathbf{r}$ in the xz plane, then it has coordinates $\mathbf{r}=x\hat{\mathbf{x}}+z\hat{\mathbf{z}}$. Then the Biot-Savart law says that the contribution to the magnetic field from a line element dl' at point $\mathbf{r}'$ is

$$d\mathbf{B}=\frac{\mu_0}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|^3}\mathbf{I}\times\left(\mathbf{r}-\mathbf{r}'\right)dl' \tag{3}$$

The cross product is what interests us, and we get for the current element under consideration

$$\mathbf{I}\times\left(\mathbf{r}-\mathbf{r}_1'\right)=\left[I_x\hat{\mathbf{x}}+I_z\hat{\mathbf{z}}\right]\times\left[\left(x-x'\right)\hat{\mathbf{x}}-y'\hat{\mathbf{y}}+\left(z-z'\right)\hat{\mathbf{z}}\right] \tag{4}$$

$$=y'I_z\hat{\mathbf{x}}-\left[I_x\left(z-z'\right)-I_z\left(x-x'\right)\right]\hat{\mathbf{y}}-y'I_x\hat{\mathbf{z}} \tag{5}$$

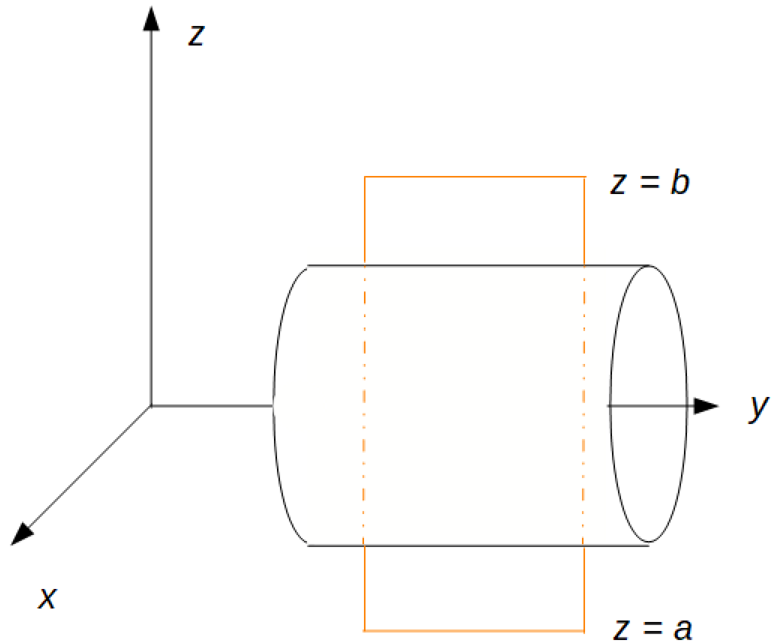

FIGURE 1. Applying Ampère's law to a solenoid. The solenoid's axis is the y axis, and the orange rectangle used to apply Ampère's law lies in the yz plane. The solenoid is shown with a circular cross section, but the argument applies to any cross section.

Because of the symmetry of the solenoid, there is a point on the solenoid at position $\mathbf{r}_2' = x'\hat{\mathbf{x}} - y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$ which has the same current and (since $\mathbf{r}$ has no y component) the same value of $|\mathbf{r}-\mathbf{r}'|$. Thus the only difference is that the sign of y' has changed, so we get

$$\mathbf{I}\times\left(\mathbf{r}-\mathbf{r}_2'\right) = -y'I_z\hat{\mathbf{x}} - \left[I_x\left(z-z'\right) - I_z\left(x-x'\right)\right]\hat{\mathbf{y}} + y'I_x\hat{\mathbf{z}} \qquad (6)$$

When we add the contributions from these symmetric elements, only the y component is left, so we conclude that the field must be entirely in the y direction.

The magnitude of the field could be worked out from the Biot-Savart law if we knew the shape of the cross-section, but using Ampère's law, we can choose a rectangular loop lying in the yz plane as in Fig. 1. The two sides of the loop parallel to the y axis have unit length. If this loop has both edges outside the solenoid, with the lower edge in the plane $z=a$ and the upper edge at $z=b$, then there is no net enclosed current, since the current flowing through the loop at the $z=b$ end is equal and opposite to the current flowing at the $z=a$ end. The contribution from the two edges parallel to the z axis cancel (the path along one of these edges is equal and opposite to the path along the other), so we get

$$\oint \mathbf{B}\cdot d\mathbf{l} = B(b)-B(a) \quad (7)$$

$$= 0 \quad (8)$$

That is, the field outside the solenoid is constant. At this point, the assumption is made that the field must go to zero at infinite distance (something that doesn't sit quite right, since we are dealing with an infinite solenoid after all, but anyway) so the field must be zero everywhere outside the solenoid.

Inside the solenoid, we could set up another loop that doesn't cross the current and conclude again that the field inside is also constant. To get its magnitude, we use a loop that *does* cross the current, with one edge inside and one outside (that is, we could move the $z=a$ side of the rectangle in Fig. 1 up so that it lies inside the solenoid). The enclosed current for a unit length is nI and the only edge that contributes to the line integral is the edge inside the solenoid so we have

$$\oint \mathbf{B}\cdot d\mathbf{l} = B_{in} \quad (9)$$

$$= \mu_0 nI \quad (10)$$

Thus the magnetic field inside an infinite solenoid is independent of the cross-sectional shape.

Using a similar technique, it is possible to show that the magnetic field inside a torus-shaped solenoid (again, the cross-section doesn't matter) is

$$B_{torus} = \frac{\mu_0 NI}{2\pi s} \quad (11)$$

where s is the radial distance from the axis of the torus, if this places the observation point inside the torus (the field is again zero outside). In this case, the field inside is not constant, but falls off linearly with distance from the axis. For a torus with a radius large compared with its cross-sectional dimensions, we can take s to be essentially 'the' radius of the torus, so that $N/2\pi s=n$ and the formula reduces to that for a linear solenoid.

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