SPHERICAL ELECTROMAGNETIC WAVE

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One form of spherical wave has an electric component given by

$$\mathbf{E}(r,\theta\phi,t)=A\frac{\sin\theta}{r}\left[\cos(kr-\omega t)-\frac{\sin(kr-\omega t)}{kr}\right]\hat{\boldsymbol{\phi}} \tag{1}$$

where A is a constant. We can derive the corresponding magnetic field from Maxwell's equations in vacuum:

$$\nabla\cdot\mathbf{E}=0 \tag{2}$$
$$\nabla\cdot\mathbf{B}=0 \tag{3}$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{4}$$
$$\nabla\times\mathbf{B}=\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{5}$$

The calculations are straightforward but can get messy, so we'll use Maple to do the derivatives. We get

$$\begin{aligned}\nabla\times\mathbf{E}&=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right)\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\hat{\boldsymbol{\theta}}\\&=\frac{2A\cos\theta}{r^2}\left[\cos(kr-\omega t)-\frac{\sin(kr-\omega t)}{kr}\right]\hat{\mathbf{r}}+\\&\quad\frac{A\sin\theta}{r}\left[\left(k-\frac{1}{kr^2}\right)\sin(kr-\omega t)+\frac{\cos(kr-\omega t)}{r}\right]\hat{\boldsymbol{\theta}}\end{aligned} \tag{6}$$

Integrating this with respect to t we get

$$\begin{aligned}\mathbf{B}=&\frac{2A\cos\theta}{r^2\omega}\left[\sin(kr-\omega t)+\frac{\cos(kr-\omega t)}{kr}\right]\hat{\mathbf{r}}+\\&\frac{A\sin\theta}{r^3\omega}\left[\left(\frac{1}{k}-kr^2\right)\cos(kr-\omega t)+r\sin(kr-\omega t)\right]\hat{\boldsymbol{\theta}}\end{aligned} \tag{7}$$

We can verify that $\mathbf{E}$ and $\mathbf{B}$ satisfy the other 3 Maxwell equations by direct calculation.

$$\nabla\cdot\mathbf{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\right) = 0 \tag{8}$$

$$\nabla\cdot\mathbf{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta B_\theta\right) \tag{9}$$

$$= \frac{2A\cos\theta}{r^2\omega}\left[\left(k - \frac{1}{kr^2}\right)\cos\left(kr-\omega t\right) - \frac{\sin\left(kr-\omega t\right)}{r}\right] +$$

$$\frac{2A\cos\theta}{r^2\omega}\left[\frac{1}{r^2}\left(\frac{1}{k} - kr^2\right)\cos\left(kr-\omega t\right) + \frac{\sin\left(kr-\omega t\right)}{r}\right]$$

$$= 0 \tag{10}$$

$$\nabla\times\mathbf{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rB_\theta\right) - \frac{\partial B_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \tag{11}$$

$$= -\frac{2A\sin\theta}{\omega r^3}\left(\left(\frac{1}{kr} - kr\right)\cos\left(kr-\omega t\right) + \sin\left(kr-\omega t\right)\right)\hat{\boldsymbol{\phi}} +$$

$$\frac{A\sin\theta}{\omega r^3}\left[-kr\cos\left(kr-\omega t\right) - \left(\frac{1}{k} - kr^2 - 1\right)\sin\left(kr-\omega t\right)\right]\hat{\boldsymbol{\phi}} +$$

$$\frac{2A\sin\theta}{\omega r^3}\left[\sin\left(kr-\omega t\right) + \frac{\cos\left(kr-\omega t\right)}{kr}\right]\hat{\boldsymbol{\phi}}$$

$$= A\frac{\sin\theta}{r^2 c}\left[kr\sin\left(kr-\omega t\right) + \cos\left(kr-\omega t\right)\right]\hat{\boldsymbol{\phi}} \tag{12}$$

where in the last line we used $\omega/k = c$ and collected terms. From 1 we get

$$\frac{\partial\mathbf{E}}{\partial t} = A\frac{\sin\theta}{r}\left[\omega\sin\left(kr-\omega t\right) + \frac{\omega\cos\left(kr-\omega t\right)}{kr}\right]\hat{\boldsymbol{\phi}} \tag{13}$$

$$= A\frac{\sin\theta}{r^2}\left[krc\sin\left(kr-\omega t\right) + c\cos\left(kr-\omega t\right)\right]\hat{\boldsymbol{\phi}} \tag{14}$$

$$= c^2\nabla\times\mathbf{B} \tag{15}$$

so the final Maxwell equation is satisfied.

The Poynting vector is, after simplifying

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{16}$$

$$= \hat{\mathbf{r}}\frac{A^2\sin^2\theta}{\mu_0\omega kr^5}\left[\left(\frac{1}{2} - r^2k^2\right)\sin\left[2\left(kr-\omega t\right)\right] - kr\cos\left[2\left(kr-\omega t\right)\right] + k^3r^3\cos^2\left(kr-\omega t\right)\right] +$$

$$\hat{\boldsymbol{\theta}}\frac{A^2\sin 2\theta}{\mu_0\omega kr^5}\left[\frac{1}{2}\left(k^2r^2 - 1\right)\sin\left[2\left(kr-\omega t\right)\right] + kr\cos\left[2\left(kr-\omega t\right)\right]\right] \tag{17}$$

The intensity, or time average of the Poynting vector is

$$\mathbf{I}=\langle\mathbf{S}\rangle=\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\mathbf{S}dt \tag{18}$$

$$=\frac{A^2k\sin^2\theta}{2\mu_0\omega r^2}\hat{\mathbf{r}} \tag{19}$$

The energy flows radially outwards and falls off as r^{-2}.

The total power radiated is the integral of $\mathbf{I}\cdot d\mathbf{a}$ over a sphere:

$$P=\int\mathbf{I}\cdot d\mathbf{a} \tag{20}$$

$$=\pi\frac{A^2k}{\mu_0\omega}\int_0^{\pi}\frac{r^2}{r^2}\sin^3\theta d\theta \tag{21}$$

$$=\frac{4\pi A^2}{3\mu_0}\frac{k}{\omega} \tag{22}$$

$$=\frac{4\pi A^2}{3\mu_0 c} \tag{23}$$

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