SURFACE CHARGES

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Post date: 1 June 2021.

A few examples of fields due to surface charge.

Example 1. First, we consider a circular disk of radius R with surface charge density σ lying in the xy plane and centred at the origin. Find the electric field at a point on the z axis.

To solve this we can make use of the solution to the circular loop. In this case we're considering a circular ring of circumference $2\pi r$ and thickness dr, so the amount of charge in the ring is $2\pi r\sigma dr$ and from the earlier solution, the field due to this ring is

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\pi r\sigma z}{(z^2+r^2)^{3/2}}dr \tag{1}$$

To get the total field from the disk, we integrate over r:

$$E = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\int_0^R \frac{r}{(z^2+r^2)^{3/2}}dr \tag{2}$$

$$= \frac{1}{4\pi\epsilon_0}2\pi\sigma\frac{\left(\sqrt{z^2+R^2}-z\right)}{\sqrt{z^2+R^2}} \tag{3}$$

To get the limiting behaviours we can Taylor-expand the result. For $z \gg R$, we expand about $R=0$ and find the leading non-zero term is in R^2:

$$E \to \frac{1}{4\pi\epsilon_0}2\pi\sigma\frac{1}{2z^2}R^2 \tag{4}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\pi\sigma R^2}{z^2} \tag{5}$$

This is correct since the total charge on the disk is $\sigma\pi R^2$ so the field is that due to a point charge of that amount.

If we let $R \to \infty$ we get

$$E \rightarrow \frac{1}{4\pi\epsilon_0} 2\pi\sigma \tag{6}$$
$$= \frac{\sigma}{2\epsilon_0} \tag{7}$$

This is the field due to an infinite plane of charge. Note that the field is independent of z so is the same no matter how far away from the plane we are.

Example 2. We have a spherical shell of charge with radius R and surface density σ, centred at the origin. Again, we seek the field at a point on the z axis.

Using spherical coordinates, a point on the sphere has coordinates (R, θ, ϕ) where θ is the angle from the positive z axis, and ϕ is the azimuthal angle. We can use the cosine law to write the distance between a point on the sphere and the field point:

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{z^2 + R^2 - 2zR\cos\theta} \tag{8}$$

By symmetry, the field will again be in the z direction, so we need the z component of $\mathbf{r} - \mathbf{r}'$. To get this, we need the angle α between $\mathbf{r} - \mathbf{r}'$ and the z axis. To get this, project the point on the sphere onto the z axis; this gives a point with z coordinate $R\cos\theta$. The remaining distance along the z axis to the field point is therefore $z - R\cos\theta$, but this distance is the projection of $\mathbf{r} - \mathbf{r}'$ onto the z axis. The cosine of the angle between $\mathbf{r} - \mathbf{r}'$ and the z axis is this projection divided by $|\mathbf{r} - \mathbf{r}'|$, so we get

$$\cos\alpha = \frac{z - R\cos\theta}{|\mathbf{r} - \mathbf{r}'|} \tag{9}$$
$$= \frac{z - R\cos\theta}{\sqrt{z^2 + R^2 - 2zR\cos\theta}} \tag{10}$$

Now for a given value of θ, we have a ring of charge with radius $R\sin\theta$ and thickness $Rd\theta$ at z distance $|\mathbf{r} - \mathbf{r}'|$ from the field point, so we can integrate over θ to get the total field.

$$E = \frac{\sigma}{4\pi\epsilon_0} \int_0^{\pi} \frac{(z - R\cos\theta)(2\pi R\sin\theta)(Rd\theta)}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} \tag{11}$$

This integral can be done using Maple, but there are two possibilities. First, if $z > R$ so the field point is outside the sphere, we get

$$E = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma}{z^2} \tag{12}$$

Since $4\pi R^2\sigma$ is the total charge on the sphere, we see that the sphere behaves like a point charge for all field points outside it.

Second, if $z < R$ so we are inside the sphere, we get

$$E = 0 \tag{13}$$

So anywhere inside a spherical shell with a uniform charge distribution, we feel no field at all.

Example 3

The result of the last example can be used to find the field due to a sphere that contains a uniform volume charge density ρ. Since each spherical shell within the sphere behaves as a point charge to all points outside the shell, the field at a point outside the sphere ($z > R$) is just

$$E = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3\rho}{3z^2} \tag{14}$$

At a point inside the sphere, all shells outside the field point contribute nothing, so we get, for $z < R$:

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0}\frac{4\pi z^3\rho}{3z^2} && (15)\\ &= \frac{z\rho}{3\epsilon_0} && (16) \end{aligned}$$

The field thus increases linearly within the sphere and then falls off as an inverse square outside.