SYNCHROTRON RADIATION

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Post date: 21 August 2021.

One common instance of an accelerated charge is a charge moving in a circle. In this case the particle's instantaneous velocity $\mathbf{v}$ is always perpendicular to its instantaneous acceleration $\mathbf{a}$. This is known as *synchrotron radiation*, since it is the radiation given off by particles in a synchrotron particle accelerator, where charged particles move in circular orbits between the poles of a magnet.

We can use the Liénard formula to work out the power radiated by such a charge:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c^2} \frac{\left|\hat{\mathbf{r}} \times (\mathbf{u} \times \mathbf{a})\right|^2}{(\hat{\mathbf{r}} \cdot \mathbf{u})^5} \tag{1}$$

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left[a^2 - \frac{|\mathbf{v} \times \mathbf{a}|^2}{c^2}\right] \tag{2}$$

At one instant of time, we can take

$$\mathbf{v} = v\hat{\mathbf{z}} \tag{3}$$

$$\mathbf{a} = a\hat{\mathbf{x}} \tag{4}$$

$$\hat{\mathbf{r}} = s_\theta c_\phi \hat{\mathbf{x}} + s_\theta s_\phi \hat{\mathbf{y}} + c_\theta \hat{\mathbf{z}} \tag{5}$$

$$\mathbf{u} = c\hat{\mathbf{r}} - \mathbf{v} \tag{6}$$

where we're using our usual shorthand for trig functions: $s_\theta \equiv \sin\theta$, $c_\theta \equiv \cos\theta$ and so on. We can now work out the components of 1:

$$\hat{\mathfrak{r}}\cdot\mathbf{u} = c\hat{\mathfrak{r}}\cdot\hat{\mathfrak{r}} - vc_\theta \tag{7}$$
$$= c - vc_\theta \tag{8}$$
$$= c\left(1-\beta c_\theta\right) \tag{9}$$
$$\mathbf{u}\times\mathbf{a} = c\hat{\mathfrak{r}}\times a\hat{\mathbf{x}} - v\hat{\mathbf{z}}\times a\hat{\mathbf{x}} \tag{10}$$
$$= ca\left(-s_\theta s_\phi\hat{\mathbf{z}} + c_\theta\hat{\mathbf{y}}\right) - av\hat{\mathbf{y}} \tag{11}$$
$$\hat{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ s_\theta c_\phi & s_\theta s_\phi & c_\theta \\ 0 & cac_\theta - av & -cas_\theta s_\phi \end{vmatrix} \tag{12}$$
$$= \hat{\mathbf{x}}\left[-cas_\theta^2 s_\phi^2 - c_\theta\left(cac_\theta - av\right)\right] + \tag{13}$$
$$\hat{\mathbf{y}}cas_\theta^2 s_\phi c_\phi + \hat{\mathbf{z}}\left(cas_\theta c_\theta c_\phi - avs_\theta c_\phi\right)$$

Taking the square of this last vector leads to a lengthy expression which can be simplified by applying $s^2+c^2=1$ repeatedly. We get, using $\beta = v/c$:

$$\frac{1}{a^2c^2}\left|\hat{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a})\right|^2 = s_\theta^4 c_\phi^4 + 1 + c_\theta^2\beta^2 - 2s_\theta^2 c_\phi^2 + 2s_\theta^2 c_\phi^2 c_\theta\beta - 2c_\theta\beta + \tag{14}$$
$$s_\theta^4 s_\phi^2 c_\phi^2 + s_\theta^2 c_\phi^2 c_\theta^2 - 2s_\theta^2 c_\phi^2 c_\theta\beta + s_\theta^2 c_\phi^2\beta^2$$

We can simplify this as follows. The first and seventh terms combine to give

$$s_\theta^4 c_\phi^4 + s_\theta^4 s_\phi^2 c_\phi^2 = s_\theta^4 c_\phi^2\left(c_\phi^2 + s_\phi^2\right) = s_\theta^4 c_\phi^2 \tag{15}$$

Combining this with the eighth term:

$$s_\theta^4 c_\phi^2 + s_\theta^2 c_\phi^2 c_\theta^2 = s_\theta^2 c_\phi^2\left(s_\theta^2 + c_\theta^2\right) = s_\theta^2 c_\phi^2 \tag{16}$$

Combining this with the fourth and last terms we get

$$s_\theta^2 c_\phi^2 - 2s_\theta^2 c_\phi^2 + s_\theta^2 c_\phi^2\beta^2 = -\left(1-\beta^2\right)s_\theta^2 c_\phi^2 \tag{17}$$

The second, third and sixth terms combine to give

$$1 + c_\theta^2\beta^2 - 2c_\theta\beta = \left(1-\beta c_\theta\right)^2 \tag{18}$$

Finally, the fifth and ninth terms cancel, so we're left with

$$\frac{1}{a^2c^2}\left|\hat{\mathfrak{r}}\times(\mathbf{u}\times\mathbf{a})\right|^2 = \left(1-\beta c_\theta\right)^2 - \left(1-\beta^2\right)s_\theta^2 c_\phi^2 \tag{19}$$

Putting everything together we get

$$\frac{dP}{d\Omega}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\left(1-\beta c_\theta\right)^2-\left(1-\beta^2\right)s_\theta^2 c_\phi^2}{\left(1-\beta c_\theta\right)^5} \tag{20}$$

To get the total power, we need to integrate this over all solid angles, so we get

$$P=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\int_0^\pi\int_0^{2\pi} d\phi d\theta s_\theta \frac{\left(1-\beta c_\theta\right)^2-\left(1-\beta^2\right)s_\theta^2 c_\phi^2}{\left(1-\beta c_\theta\right)^5} \tag{21}$$

The integral over ϕ is easy, using

$$\int_0^{2\pi} c_\phi^2 d\phi=\pi \tag{22}$$

so we're left with the integral over θ:

$$P=\frac{\mu_0 q^2 a^2}{16\pi c}\int_0^\pi d\theta \frac{2\left(1-\beta c_\theta\right)^2-\left(1-\beta^2\right)s_\theta^2}{\left(1-\beta c_\theta\right)^5}s_\theta \tag{23}$$

$$=\frac{\mu_0 q^2 a^2}{16\pi c}\int_0^\pi d\theta \frac{2\left(1-\beta c_\theta\right)^2-\left(1-\beta^2\right)\left(1-c_\theta^2\right)}{\left(1-\beta c_\theta\right)^5}s_\theta \tag{24}$$

This nasty looking integral can be done by using partial fractions, since it is the ratio of two polynomials in c_θ. I did the integral using Maple, but if you're interested in doing it by hand, the partial fraction decomposition is

$$\frac{2\left(1-\beta c_\theta\right)^2-\left(1-\beta^2\right)\left(1-c_\theta^2\right)}{\left(1-\beta c_\theta\right)^5}=-\frac{\left(\beta^4-2\beta^2+1\right)}{\beta^2\left(\beta\cos(\theta)-1\right)^5}+2\frac{\left(\beta^2-1\right)}{\beta^2\left(\beta\cos(\theta)-1\right)^4}-\frac{\left(\beta^2+1\right)}{\left(\beta\cos(\theta)-1\right)^3\beta^2} \tag{25}$$

The presence of the extra $\sin\theta$ from the solid angle element saves the day, since it multiplies each term in the partial fraction expansion, providing the derivative of $\cos\theta$ on the top of each fraction. For example

$$\int d\theta \frac{\sin\theta}{\left(\beta\cos(\theta)-1\right)^5}=\frac{1}{4\beta\left(\beta\cos(\theta)-1\right)^4} \tag{26}$$

with the other two terms having similar integrals.

The result of the integral is

$$\int_0^\pi d\theta \frac{2\left(1-\beta c_\theta\right)^2-\left(1-\beta^2\right)\left(1-c_\theta^2\right)}{\left(1-\beta c_\theta\right)^5} s_\theta = \frac{8}{3\left(1-\beta\right)^2\left(1+\beta\right)^2} \tag{27}$$

$$= \frac{8}{3\left(1-\beta^2\right)^2} \tag{28}$$

$$= \frac{8\gamma^4}{3} \tag{29}$$

so we get for the total power

$$P = \frac{\mu_0 q^2 a^2}{16\pi c}\frac{8\gamma^4}{3} \tag{30}$$

$$= \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c} \tag{31}$$