TORQUE ON A MAGNETIC MOMENT - GENERAL CURRENT DISTRIBUTION

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.
Post date: 19 Mar 2021.
The torque on a square current loop in a constant, uniform magnetic field is given by

$$\mathbf{N}=\mathbf{m}\times\mathbf{B} \tag{1}$$

It turns out that this formula applies to any current distribution, although the proof is a bit tricky. Since the magnetic moment for a collection of line currents is defined as

$$\mathbf{m}=I\mathbf{a}=\frac{1}{2}I\int\mathbf{r}\times d\mathbf{l} \tag{2}$$

the natural generalization of magnetic moment to a general volume current density is

$$\mathbf{m}=\frac{1}{2}\int\mathbf{r}\times\mathbf{J}d^3\mathbf{r} \tag{3}$$

The Lorentz force law for a volume current is

$$\mathbf{F}=\int\mathbf{J}\times\mathbf{B}d^3\mathbf{r} \tag{4}$$

so the torque is

$$\mathbf{N}=\int\mathbf{r}\times(\mathbf{J}\times\mathbf{B})\,d^3\mathbf{r} \tag{5}$$

We can convert this using a vector identity:

$$\mathbf{N}=\int(\mathbf{r}\cdot\mathbf{B})\,\mathbf{J}d^3\mathbf{r}-\int(\mathbf{r}\cdot\mathbf{J})\,\mathbf{B}d^3\mathbf{r} \tag{6}$$

To proceed further, we need another vector identity, which states that for scalar fields f and g and localized vector field $\mathbf{J}$, we have

$$\int[f\mathbf{J}\cdot\nabla g+g\mathbf{J}\cdot\nabla f+fg\nabla\cdot\mathbf{J}]\,d^3\mathbf{r}=0 \tag{7}$$

This follows, since if we integrate the middle term by parts, we get

$$\int g\mathbf{J}\cdot\nabla f = fg\left(J_x+J_y+J_z\right)-\int f\nabla\cdot\left(g\mathbf{J}\right)d^3\mathbf{r} \tag{8}$$

$$= 0-\int\left[f\mathbf{J}\cdot\nabla g+fg\nabla\cdot\mathbf{J}\right]d^3\mathbf{r} \tag{9}$$

where the integrated term is zero if we assume that **J** goes to zero at infinity (that is, it's localized).
Now if we consider steady currents, then $\nabla\cdot\mathbf{J}=0$ so the identity reduces to

$$\int\left[f\mathbf{J}\cdot\nabla g+g\mathbf{J}\cdot\nabla f\right]d^3\mathbf{r}=0 \tag{10}$$

If we take $f=g=r=\sqrt{x^2+y^2+z^2}$, then $\nabla r=\mathbf{r}/r$ and

$$2\int r\mathbf{J}\cdot\frac{\mathbf{r}}{r}d^3\mathbf{r}=2\int\mathbf{r}\cdot\mathbf{J}d^3\mathbf{r}=0 \tag{11}$$

so the second integral in the torque 6 is zero (remember **B** is constant, so it comes outside the integral).
For the first term, we use the identity 7 with $f=x$, $g=y$:

$$\int\left(xJ_y+yJ_x\right)d^3\mathbf{r}=0 \tag{12}$$

Choosing the other coordinates in turn, we have in general for i and j equal to any combination of x, y and z:

$$\int\left(r_iJ_j+r_jJ_i\right)d^3\mathbf{r}=0 \tag{13}$$

Now look at the ith component of the first term of 6:

$$N_i = \int\left(\mathbf{r}\cdot\mathbf{B}\right)J_id^3\mathbf{r} \tag{14}$$

$$= \sum_j B_j\int r_jJ_id^3\mathbf{r} \tag{15}$$

$$= \frac{1}{2}\sum_j B_j\int\left(r_jJ_i-r_iJ_j\right)d^3\mathbf{r} \tag{16}$$

where we used 13 to get the last line.
If we take $i=x$ for example, we get

$$N_x = \frac{1}{2} B_y \int (yJ_x - xJ_y)\, d^3\mathbf{r} + \frac{1}{2} B_z \int (zJ_x - xJ_z)\, d^3\mathbf{r} \tag{17}$$

$$= \frac{1}{2} \int \left[-(\mathbf{r}\times\mathbf{J})_z B_y + (\mathbf{r}\times\mathbf{J})_y B_z \right] d^3\mathbf{r} \tag{18}$$

$$= \frac{1}{2} \left[\left(\int \mathbf{r}\times\mathbf{J} d^3\mathbf{r} \right) \times \mathbf{B} \right]_x \tag{19}$$

The other two components work out similarly, so we get from 3

$$\mathbf{N} = \left[\frac{1}{2} \int \mathbf{r}\times\mathbf{J} d^3\mathbf{r} \right] \times \mathbf{B} \tag{20}$$

$$= \mathbf{m}\times\mathbf{B} \tag{21}$$