TWO CHARGED WIRES AND A CYLINDER

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Another example of the method of images. We are given two infinite charged wires, one with linear charge density $+\lambda$ and the other with $-\lambda$. These wires lie in the $x-y$ plane along the lines $y=\pm a$. Between them is an infinite conducting cylinder of radius $R<a$, with its axis on the x axis. The cylinder carries no net charge. We are to find the potential everywhere outside the cylinder.

The solution to this problem relies on an earlier example in which we worked out the potential due to two charged wires on their own. We saw there that the equipotential surfaces for two charged wires were circular cylinders, so we should be able to set up an image configuration in which the surface of the cylinder can be replaced by two image wires inside the cylinder.

In the charged wires problem, we had two wires at $y=\pm a$ and found that the potential could be written as

$$V=\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(y+a)^2+z^2}{(y-a)^2+z^2} \tag{1}$$

We also found that for a constant potential K, if we define the value $A\equiv e^{4\pi\epsilon_0 K/\lambda}$ we could get expressions for the radius R and axis y_0 of the equipotential cylinders:

$$y_0 = \frac{A+1}{A-1}a \tag{2}$$

$$R = \sqrt{\frac{4a^2A}{(A-1)^2}} \tag{3}$$

A useful relation between these two is

$$y_0^2-R^2=a^2 \tag{4}$$

To apply the method of images and the above solution, we need to create one image for each wire. Consider first the wire on the left, at $y=-a$. We want to create an image of this wire at $y=-b$, where this location (to be determined) is inside the cylinder. We can map this image problem onto the two-wire problem by noting that the midpoint between the wire and its image is at $y = -(a+b)/2$. If we define

$$y_1 \equiv y + b + (a-b)/2 \tag{5}$$

then we have

$$y_1 = \begin{cases} \frac{a-b}{2} & \text{when } y = -b \\ -\frac{a-b}{2} & \text{when } y = -a \end{cases} \tag{6}$$

Thus the potential due to the real wire at $y = -a$ and its image at $y = -b$ can be found by using y_1 as the variable in 1 and replacing a by $\frac{a-b}{2}$ in 1. This gives us

$$V_1 = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{(y_1 + \frac{a-b}{2})^2 + z^2}{(y_1 - \frac{a-b}{2})^2 + z^2} \tag{7}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{(y+a)^2 + z^2}{(y+b)^2 + z^2} \tag{8}$$

On the other side, the wire at $y = +a$ has an image at $y = +b$. Defining

$$y_2 \equiv y - b - (a-b)/2 \tag{9}$$

we get for the potential of this pair:

$$V_2 = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{(y_2 + \frac{a-b}{2})^2 + z^2}{(y_2 - \frac{a-b}{2})^2 + z^2} \tag{10}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{(y-b)^2 + z^2}{(y-a)^2 + z^2} \tag{11}$$

The only remaining problem is to find b. We can use 4 for this. Consider the first wire-image pair. The distance y_0 is the distance from the midpoint between the wire and image to the axis of the cylinder, which is $y_0 = \frac{a-b}{2} + b = \frac{a+b}{2}$. The quantity a in this relation is half the distance between the wire and its image, so in this case must be replaced by $\frac{a-b}{2}$. We therefore have

$$\left(\frac{a+b}{2}\right)^2 - R^2 = \left(\frac{a-b}{2}\right)^2 \tag{12}$$

$$b = \frac{R^2}{a} \tag{13}$$

Since potentials obey the superposition principle, we can just add the two potentials from the two image-wire pairs to get the total potential:

$$V = V_1 + V_2 \tag{14}$$
$$= \frac{\lambda}{4\pi\epsilon_0}\left[\ln\frac{(y+a)^2+z^2}{(y+b)^2+z^2} + \ln\frac{(y-b)^2+z^2}{(y-a)^2+z^2}\right] \tag{15}$$
$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2+z^2}{(y+b)^2+z^2}\cdot\frac{(y-b)^2+z^2}{(y-a)^2+z^2}\right] \tag{16}$$
$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2+z^2}{(y+R^2/a)^2+z^2}\cdot\frac{(y-R^2/a)^2+z^2}{(y-a)^2+z^2}\right] \tag{17}$$

To convert this to cylindrical coordinates, we can use $y^2+z^2=r^2$ and $y=r\cos\phi$:

$$V(r,\phi) = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{r^2+a^2+2ar\cos\phi}{r^2+2R^2r\cos\phi/a+R^4/a^2}\cdot\frac{r^2-2R^2r\cos\phi/a+R^4/a^2}{r^2+a^2-2ar\cos\phi}\right] \tag{18}$$
$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{r^2+a^2+2ar\cos\phi}{a^2r^2/R^2+2ar\cos\phi+R^2}\cdot\frac{a^2r^2/R^2-2ar\cos\phi+R^2}{r^2+a^2-2ar\cos\phi}\right] \tag{19}$$

Note that if $r=R$, then the argument of the logarithm becomes 1, so $V=0$ on the surface of the cylinder. This happens even though the potential of each wire-image pair separately is *not* zero at $r=R$. In fact, for the first pair, we have

$$V_1(R,\phi) = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{a^2}{R^2} \tag{20}$$

and for the second pair, we get

$$V_2(R,\phi) = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{a^2} \tag{21}$$

Also, if $\phi=\pi/2$ or $3\pi/2$, $V=0$. This corresponds to the xz plane, which is the plane of symmetry in the problem.