UNIQUENESS OF POTENTIAL IN DIELECTRICS

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When we considered electric fields in a vacuum, we found that if we specify the charge distribution in some region $\mathcal{V}$, and also specify either the potential or its normal derivative on the boundary of that region, then the potential inside $\mathcal{V}$ is unique. Here, we'll show that uniqueness of the potential also applies to the case where $\mathcal{V}$ contains some linear dielectric. The assumptions we make are:

(1) The potential V is specified on all boundaries of $\mathcal{V}$.
(2) The free charge distribution ρ_f is specified everywhere within $\mathcal{V}$.
(3) The distribution of dielectric within $\mathcal{V}$ is fixed, and all dielectric constants are specified.

The proof follows a similar line of reasoning to that used in the electric field case. As before we'll suppose that there are two distinct potentials V_1 and V_2 that satisfy the conditions. We'll also define the displacements due to these potentials as $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. Now we consider the difference between these two solutions, so we have $V_3 \equiv V_1 - V_2$ and $\boldsymbol{D}_3 \equiv \boldsymbol{D}_1 - \boldsymbol{D}_2$. Now we look at this volume integral and convert it to a surface integral in the usual way:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \boldsymbol{D}_3) \, d^3\mathbf{r} \;=\; \int_{\mathcal{S}} V_3 \boldsymbol{D}_3 \cdot d\mathbf{a} \tag{1}$$

By assumption, on the surface $\mathcal{S}$, $V_3 = V_1 - V_2 = 0$ since the potential is specified everywhere on the boundary. Therefore:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \boldsymbol{D}_3) \, d^3\mathbf{r} = 0 \tag{2}$$

Now we expand the integrand using a standard theorem from vector calculus:

$$\nabla \cdot (V_3 \boldsymbol{D}_3) = \boldsymbol{D}_3 \cdot \nabla V_3 + V_3 \nabla \cdot \boldsymbol{D}_3 \tag{3}$$

From the formula for the divergence of the displacement $\nabla \cdot \boldsymbol{D} = \rho_f$, and the fact that the free charge density ρ_f is specified everywhere:

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$$\nabla \cdot \boldsymbol{D}_3 = \nabla \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) \tag{4}$$

$$= \rho_f - \rho_f \tag{5}$$

$$= 0 \tag{6}$$

Therefore we have

$$\nabla \cdot (V_3 \boldsymbol{D}_3) = \boldsymbol{D}_3 \cdot \nabla V_3 \tag{7}$$

For a linear dielectric, $\boldsymbol{D} = \epsilon \mathbf{E}$ and in general, $\mathbf{E} = -\nabla V$, so

$$\nabla \cdot (V_3 \boldsymbol{D}_3) = \boldsymbol{D}_3 \cdot \nabla V_3 \tag{8}$$

$$= -\epsilon E_3^2 \tag{9}$$

$$= -\epsilon |\mathbf{E}_1 - \mathbf{E}_2|^2 \tag{10}$$

Thus the volume integral becomes

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \boldsymbol{D}_3) \, d^3 \mathbf{r} = -\int_{\mathcal{V}} \epsilon |\mathbf{E}_1 - \mathbf{E}_2|^2 \, d^3 \mathbf{r} \tag{11}$$

$$= 0 \tag{12}$$

(Note that we can't take ϵ outside the integral since in general it varies over the volume, depending on what dielectrics are present.)

Now the integrand is non-negative everywhere, since the permittivity $\epsilon \geq \epsilon_0$ so the only way the integral can be zero is if $\mathbf{E}_1 = \mathbf{E}_2$ everywhere inside $\mathcal{V}$. This means that $V_1 - V_2 = k$ for some constant k, but since $V_1 = V_2$ on the boundary, we must have $k = 0$ and $V_1 = V_2$ everywhere. QED.