# VOLUME CURRENT DENSITY AND DIPOLE MOMENT

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There is a relation between the volume current density and the dipole moment. Suppose we have a collection of charges and currents contained within a finite volume $\mathcal{V}$. Consider the integral

$$\int_{\mathcal{V}} \nabla\cdot(x\mathbf{J})\,d^3\mathbf{r} \tag{1}$$

Looking at the integrand, we have

$$\begin{aligned}\nabla\cdot(x\mathbf{J}) &= \nabla x\cdot\mathbf{J}+x\nabla\cdot\mathbf{J} &(2)\\ &= \hat{\mathbf{x}}\cdot\mathbf{J}+x\nabla\cdot\mathbf{J} &(3)\\ &= J_x+x\nabla\cdot\mathbf{J} &(4)\end{aligned}$$

The current density represents the rate of flow of charge across a surface $\mathcal{S}$, so the total rate of change of charge must be equal to the net flow across the surface. That is

$$\int_{\mathcal{S}}\mathbf{J}\cdot d\mathbf{a} = -\frac{d}{dt}\int_{\mathcal{V}}\rho d^3\mathbf{r} \tag{5}$$

where the minus sign indicates that a net flow of charge out of the volume (indicated by the integral on the LHS being positive) means a decrease in the amount of charge inside the volume.

By the divergence theorem, the integral on the LHS can be converted into a volume integral, so we get

$$\begin{aligned}\int_{\mathcal{S}}\mathbf{J}\cdot d\mathbf{a} &= \int_{\mathcal{V}}\nabla\cdot\mathbf{J}d^3\mathbf{r} &(6)\\ &= -\int_{\mathcal{V}}\frac{\partial\rho}{\partial t}d^3\mathbf{r} &(7)\end{aligned}$$

where we've taken the time derivative inside the integral, and converted it to a partial derivative since $\rho$ depends on space as well.

This relation has to be true for any volume, so we can equate the integrands to get

$$\boxed{\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}} \tag{8}$$

This is the differential form of the law of charge conservation.

Returning to the first equation, we can now say:

$$\nabla\cdot(x\mathbf{J}) = J_x + x\nabla\cdot\mathbf{J} \tag{9}$$

$$= J_x - x\frac{\partial\rho}{\partial t} \tag{10}$$

Integrating the LHS and using the divergence theorem in the opposite direction, we get

$$\int_{\mathcal{V}}\nabla\cdot(x\mathbf{J})\,d^3\mathbf{r}=\int_{\mathcal{S}}x\mathbf{J}\cdot d\mathbf{a} \tag{11}$$

Now if we take the bounding surface $\mathcal{S}$ to be outside all the charges and currents, then the RHS is zero, so we get

$$\int_{\mathcal{V}}J_x d^3\mathbf{r}=\int_{\mathcal{V}}x\frac{\partial\rho}{\partial t}d^3\mathbf{r} \tag{12}$$

Clearly we can write similar equations for the $y$ and $z$ components and then multiply each equation by the corresponding unit vector and add all three up to get

$$\int_{\mathcal{V}}\mathbf{J}d^3\mathbf{r}=\int_{\mathcal{V}}\mathbf{r}\frac{\partial\rho}{\partial t}d^3\mathbf{r} \tag{13}$$

The integral on the RHS is

$$\int_{\mathcal{V}}\mathbf{r}\frac{\partial\rho}{\partial t}d^3\mathbf{r}=\frac{d\mathbf{p}}{dt} \tag{14}$$

where **p** is the total dipole moment of the charge in the volume. We therefore get the relation between current density and dipole moment:

$$\boxed{\int_{\mathcal{V}}\mathbf{J}d^3\mathbf{r}=\frac{d\mathbf{p}}{dt}} \tag{15}$$

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