WAVE EQUATION - DERIVATION AND EXAMPLES

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As a prelude to the study of electromagnetic waves, we'll have a look at a derivation of the wave equation.

Waves can appear in any form of matter, as well as in electromagnetic fields, so we'll look at the easiest case for a derivation. Suppose we have a string under a tension T. Initially, we'll stretch out the string along the z axis so that it's perfectly straight. Now suppose we pull the string slightly to one side (in the x direction, say) so that the string now follows a slightly curved path. Suppose that at position z the tangent to the string makes an angle θ with the z axis and at a slightly different position $z+dz$ the angle is $\theta+d\theta$. The tension can be resolved into an x component (perpendicular to the original orientation of the string) and a z component (parallel to the original orientation). We get, at position z:

$$\begin{aligned} T_x(z) &= T\sin\theta && (1) \\ T_z(z) &= T\cos\theta && (2) \end{aligned}$$

and at position $z+dz$:

$$\begin{aligned} T_x(z+dz) &= T\sin(\theta+d\theta) && (3) \\ T_z(z+dz) &= T\cos(\theta+d\theta) && (4) \end{aligned}$$

For small angles, the sine and tangent are equal to first order:

$$\sin\theta \approx \theta \approx \tan\theta \qquad (5)$$

and the tangent is the slope of tangent to the string. Since the string was pulled back in the x direction, x is the displacement, so

$$\tan\theta = \frac{\partial x}{\partial z} \qquad (6)$$

and the difference in the x component of the tension between the two points on the string is

$$\begin{aligned} T_x(z+dz) - T_x(z) &\approx T\tan(\theta + d\theta) - T\tan\theta & (7) \\ &= T\left(\left.\frac{\partial x}{\partial z}\right|_{z+dz} - \left.\frac{\partial x}{\partial z}\right|_{z}\right) & (8) \end{aligned}$$

The second derivative is defined as

$$\frac{\partial^2 x}{\partial z^2} = \lim_{dz\to 0}\frac{1}{dz}\left(\left.\frac{\partial x}{\partial z}\right|_{z+dz} - \left.\frac{\partial x}{\partial z}\right|_{z}\right) \tag{9}$$

so the difference in T_x is, in the limit $dz \to 0$

$$dT_x = \frac{\partial^2 x}{\partial z^2} T dz \tag{10}$$

However, from Newton's law $F = ma$, the net force on the string segment is also equal to the mass of that string segment times its acceleration. If the mass per unit length is μ, then

$$dT_x = \mu dz \frac{\partial^2 x}{\partial t^2} \tag{11}$$

Equating these last two formulas gives us the wave equation

$$\frac{\partial^2 x}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 x}{\partial t^2} \tag{12}$$

The reciprocal of the coefficient (T/μ) has the units of force divided by mass, which is velocity squared, so we can define a velocity

$$v \equiv \sqrt{\frac{T}{\mu}} \tag{13}$$

and write the wave equation as

$$\boxed{\frac{\partial^2 x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 x}{\partial t^2}} \tag{14}$$

Although we haven't shown it here, it turns out that v is the speed of propagation of the wave.

It turns out that any function of form

$$x = f(z \pm vt) \tag{15}$$

is a solution of the wave equation as can readily be seen by direct differentiation.

Example 1. With

$$f = Ae^{-b(z-vt)^2} \tag{16}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = 2bAe^{-b(z-vt)^2}\left(2bv^2t^2 - 4bzvt + 2bz^2 - 1\right) \tag{17}$$

$$\frac{\partial^2 x}{\partial t^2} = 2bAv^2e^{-b(z-vt)^2}\left(2bv^2t^2 - 4bzvt + 2bz^2 - 1\right) = v^2\frac{\partial^2 x}{\partial z^2} \tag{18}$$

so 14 is satisfied.

Example 2. With

$$f = A\sin(b(z-vt)) \tag{19}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = -Ab^2\sin(b(z-vt)) \tag{20}$$

$$\frac{\partial^2 x}{\partial t^2} = -Ab^2v^2\sin(b(z-vt)) = v^2\frac{\partial^2 x}{\partial z^2} \tag{21}$$

Example 3. With

$$f = \frac{A}{b(z-vt)^2+1} \tag{22}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = \frac{8Ab^2(z-vt)^2}{\left(b(z-vt)^2+1\right)^3} - \frac{2Ab}{\left(b(z-vt)^2+1\right)^2} \tag{23}$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{8Ab^2v^2(z-vt)^2}{\left(b(z-vt)^2+1\right)^3} - \frac{2Abv^2}{\left(b(z-vt)^2+1\right)^2} = v^2\frac{\partial^2 x}{\partial z^2} \tag{24}$$

Functions that don't have the form 15 don't satisfy the wave equation.

Example 4. With

$$f = Ae^{-b\left(bz^2+vt\right)} \tag{25}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = 2Ab^2 e^{-b\left(bz^2+vt\right)}\left(2b^2z^2-1\right) \tag{26}$$

$$\frac{\partial^2 x}{\partial t^2} = Ab^2v^2 e^{-b\left(bz^2+vt\right)} \neq v^2\frac{\partial^2 x}{\partial z^2} \tag{27}$$

Example 5. With

$$f = A\sin(bz)\cos(bvt)^3 \tag{28}$$

we have

$$\frac{\partial^2 x}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3 \tag{29}$$

$$\frac{\partial^2 x}{\partial t^2} = -3A\sin(bz)\,b^3v^3t\left(3(bvt)^3\cos(bvt)^3 + 2\sin(bvt)^3\right) \neq v^2\frac{\partial^2 x}{\partial z^2} \tag{30}$$

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