

WAVE EQUATION - DERIVATION AND EXAMPLES

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As a prelude to the study of electromagnetic waves, we'll have a look at a derivation of the wave equation.

Waves can appear in any form of matter, as well as in electromagnetic fields, so we'll look at the easiest case for a derivation. Suppose we have a string under a tension T . Initially, we'll stretch out the string along the z axis so that it's perfectly straight. Now suppose we pull the string slightly to one side (in the x direction, say) so that the string now follows a slightly curved path. Suppose that at position z the tangent to the string makes an angle θ with the z axis and at a slightly different position $z + dz$ the angle is $\theta + d\theta$. The tension can be resolved into an x component (perpendicular to the original orientation of the string) and a z component (parallel to the original orientation). We get, at position z :

$$T_x(z) = T \sin \theta \quad (1)$$

$$T_z(z) = T \cos \theta \quad (2)$$

and at position $z + dz$:

$$T_x(z + dz) = T \sin(\theta + d\theta) \quad (3)$$

$$T_z(z + dz) = T \cos(\theta + d\theta) \quad (4)$$

For small angles, the sine and tangent are equal to first order:

$$\sin \theta \approx \theta \approx \tan \theta \quad (5)$$

and the tangent is the slope of tangent to the string. Since the string was pulled back in the x direction, x is the displacement, so

$$\tan \theta = \frac{\partial x}{\partial z} \quad (6)$$

and the difference in the x component of the tension between the two points on the string is

$$T_x(z + dz) - T_x(z) \approx T \tan(\theta + d\theta) - T \tan \theta \quad (7)$$

$$= T \left(\left. \frac{\partial x}{\partial z} \right|_{z+dz} - \left. \frac{\partial x}{\partial z} \right|_z \right) \quad (8)$$

The second derivative is defined as

$$\frac{\partial^2 x}{\partial z^2} = \lim_{dz \rightarrow 0} \frac{1}{dz} \left(\left. \frac{\partial x}{\partial z} \right|_{z+dz} - \left. \frac{\partial x}{\partial z} \right|_z \right) \quad (9)$$

so the difference in T_x is, in the limit $dz \rightarrow 0$

$$dT_x = \frac{\partial^2 x}{\partial z^2} T dz \quad (10)$$

However, from Newton's law $F = ma$, the net force on the string segment is also equal to the mass of that string segment times its acceleration. If the mass per unit length is μ , then

$$dT_x = \mu dz \frac{\partial^2 x}{\partial t^2} \quad (11)$$

Equating these last two formulas gives us the wave equation

$$\frac{\partial^2 x}{\partial z^2} = \frac{\mu}{T} \frac{\partial^2 x}{\partial t^2} \quad (12)$$

The reciprocal of the coefficient (T/μ) has the units of force divided by mass, which is velocity squared, so we can define a velocity

$$v \equiv \sqrt{\frac{T}{\mu}} \quad (13)$$

and write the wave equation as

$$\boxed{\frac{\partial^2 x}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 x}{\partial t^2}} \quad (14)$$

Although we haven't shown it here, it turns out that v is the speed of propagation of the wave.

It turns out that any function of form

$$x = f(z \pm vt) \quad (15)$$

is a solution of the wave equation as can readily be seen by direct differentiation.

Example 1. With

$$f = Ae^{-b(z-vt)^2} \quad (16)$$

we have

$$\frac{\partial^2 x}{\partial z^2} = 2bAe^{-b(z-vt)^2} (2bv^2t^2 - 4bzvt + 2bz^2 - 1) \quad (17)$$

$$\frac{\partial^2 x}{\partial t^2} = 2bAv^2e^{-b(z-vt)^2} (2bv^2t^2 - 4bzvt + 2bz^2 - 1) = v^2 \frac{\partial^2 x}{\partial z^2} \quad (18)$$

so 14 is satisfied.

Example 2. With

$$f = A \sin(b(z - vt)) \quad (19)$$

we have

$$\frac{\partial^2 x}{\partial z^2} = -Ab^2 \sin(b(z - vt)) \quad (20)$$

$$\frac{\partial^2 x}{\partial t^2} = -Ab^2v^2 \sin(b(z - vt)) = v^2 \frac{\partial^2 x}{\partial z^2} \quad (21)$$

Example 3. With

$$f = \frac{A}{b(z - vt)^2 + 1} \quad (22)$$

we have

$$\frac{\partial^2 x}{\partial z^2} = \frac{8Ab^2(z - vt)^2}{(b(z - vt)^2 + 1)^3} - \frac{2Ab}{(b(z - vt)^2 + 1)^2} \quad (23)$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{8Ab^2v^2(z - vt)^2}{(b(z - vt)^2 + 1)^3} - \frac{2Abv^2}{(b(z - vt)^2 + 1)^2} = v^2 \frac{\partial^2 x}{\partial z^2} \quad (24)$$

Functions that don't have the form 15 don't satisfy the wave equation.

Example 4. With

$$f = Ae^{-b(bz^2+vt)} \quad (25)$$

we have

$$\frac{\partial^2 x}{\partial z^2} = 2Ab^2 e^{-b(bz^2+vt)} (2b^2 z^2 - 1) \quad (26)$$

$$\frac{\partial^2 x}{\partial t^2} = Ab^2 v^2 e^{-b(bz^2+vt)} \neq v^2 \frac{\partial^2 x}{\partial z^2} \quad (27)$$

Example 5. With

$$f = A \sin(bz) \cos(bvt)^3 \quad (28)$$

we have

$$\frac{\partial^2 x}{\partial z^2} = -Ab^2 \sin(bz) \cos(bvt)^3 \quad (29)$$

$$\frac{\partial^2 x}{\partial t^2} = -3A \sin(bz) b^3 v^3 t \left(3(bvt)^3 \cos(bvt)^3 + 2 \sin(bvt)^3 \right) \neq v^2 \frac{\partial^2 x}{\partial z^2} \quad (30)$$

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