WAVE EQUATION - SOLUTION BY SEPARATION OF VARIABLES

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We can use separation of variables to solve the wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{1}$$

As usual, we propose a solution of form

$$f_0(z,t) = Z(z)T(t) \tag{2}$$

Substituting into the wave equation and dividing through by ZT we get

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = \frac{1}{v^2 T}\frac{d^2 T}{dt^2} \tag{3}$$

Since the LHS depends only on z and the RHS only on t, both sides must be equal to a constant, which we can call $-k^2$. Thus

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2 \tag{4}$$

$$\frac{1}{v^2 T}\frac{d^2 T}{dt^2} = -k^2 \tag{5}$$

The general solutions are

$$Z(z) = Ae^{ikz} + Be^{-ikz} \tag{6}$$

$$T(t) = Ce^{ikvt} + De^{-ikvt} \tag{7}$$

$$= Ce^{i\omega t} + De^{-i\omega t} \tag{8}$$

where $\omega \equiv kv$. Therefore

$$f_0(z,t) = \left(Ae^{ikz} + Be^{-ikz}\right)\left(Ce^{i\omega t} + De^{-i\omega t}\right) \tag{9}$$

$$= ADe^{i(kz-\omega t)} + BCe^{-i(kz-\omega t)} + ACe^{i(kz+\omega t)} + BDe^{-i(kz+\omega t)} \tag{10}$$

The most general solution is the weighted integral of this quantity over all values of k, that is

$$f(z,t)=\int_0^\infty c(k)\left[ADe^{i(kz-\omega t)}+BCe^{-i(kz-\omega t)}+ACe^{i(kz+\omega t)}+BDe^{-i(kz+\omega t)}\right]dk \tag{11}$$

If we allow k (and therefore also ω) to take on negative and positive values, we can expand the integral to $\pm\infty$ and combine terms 1 and 2, and terms 3 and 4:

$$f(z,t)=\int_{-\infty}^{\infty}\left(A_1(k)e^{i(kz-\omega t)}+A_2(k)e^{i(kz+\omega t)}\right)dk \tag{12}$$

where we've condensed the constants A,B,C,D into the coefficients $c(k)$ to produce the new coefficients $A_{1,2}(k)$.

Technically, this is as far as we can go if we want the full complex solution, but in reality we are interested only in the real part. The real part of the first exponential is the same as the real part of the second exponential, while the imaginary parts are equal and opposite, so from a physical point of view, we can write the general solution as

$$\tilde{f}(z,t)=\int_{-\infty}^{\infty}\tilde{A}(k)e^{i(kz-\omega t)}dk \tag{13}$$

where we've added tildes to indicate that this is a physical (rather than a proper mathematical) solution, and that we should look only at the real part of $\tilde{f}$ to get the actual equation of the wave. (We could equally well have used the second exponential in our physical solution, but the first exponential is more traditional.)