

## WAVE GUIDES - DERIVATION OF THE WAVE EQUATION

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Post date: 9 Apr 2021.

A wave guide is a hollow tube which allows electromagnetic waves to travel down it. Wave guides are usually made of conductors, so we'll assume that they are made of a perfect conductor so that  $\mathbf{E} = 0$  everywhere inside the conducting boundary.

The boundary conditions implied by Maxwell's equations are

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f \quad (1)$$

$$B_1^\perp - B_2^\perp = 0 \quad (2)$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \quad (3)$$

$$\frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}} \quad (4)$$

In particular, 3 tells us that the parallel component of  $\mathbf{E}$  is zero at the boundary of the wave guide. As usual, we'll take the  $z$  axis to be parallel to wave guide's axis, so the waves have the form

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0(x, y) e^{i(kz - \omega t)} \quad (5)$$

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_0(x, y) e^{i(kz - \omega t)} \quad (6)$$

Maxwell's equations inside the guide (assumed to be a vacuum) are

$$\nabla \cdot \mathbf{E} = 0 \quad (7)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (8)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (9)$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (10)$$

Previously, we applied the divergence equations to show that the waves were transverse (no  $z$  component), but that relied on the waves being unbounded plane waves, which is not the case here. It turns out that waves in

a wave guide are not transverse in general, in that at least one of  $\mathbf{E}$  and  $\mathbf{B}$  must have a longitudinal component. We therefore write

$$\tilde{\mathbf{E}}_0(x, y) = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}} \quad (11)$$

$$\tilde{\mathbf{B}}_0(x, y) = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}} \quad (12)$$

where all the components on the RHS depend on  $x$  and  $y$ , and may be complex functions. Putting these together with 5 and 6 into 9, we get (remembering that the components do not depend on  $z$ ):

$$\begin{aligned} \nabla \times \mathbf{E} &= (\partial_y E_z - ik E_y) \hat{\mathbf{x}} + (-\partial_x E_z + ik E_x) \hat{\mathbf{y}} + (\partial_x E_y - \partial_y E_x) \hat{\mathbf{z}} \\ &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \quad (13)$$

$$= i\omega B_x \hat{\mathbf{x}} + i\omega B_y \hat{\mathbf{y}} + i\omega B_z \hat{\mathbf{z}} \quad (15)$$

Equating components, we get

$$\partial_y E_z - ik E_y = i\omega B_x \quad (16)$$

$$-\partial_x E_z + ik E_x = i\omega B_y \quad (17)$$

$$\partial_x E_y - \partial_y E_x = i\omega B_z \quad (18)$$

We can apply exactly the same procedure to 10 to get the analogous equations

$$\partial_y B_z - ik B_y = -i\frac{\omega}{c^2} E_x \quad (19)$$

$$-\partial_x B_z + ik B_x = -i\frac{\omega}{c^2} E_y \quad (20)$$

$$\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2} E_z \quad (21)$$

We can solve these 6 equations to get the  $x$  and  $y$  components in terms of the  $z$  components. For example, multiplying 17 through by  $k$  and 19 through by  $\omega$  and adding, we get

$$-k\partial_x E_z + ik^2 E_x - i\frac{\omega^2}{c^2} E_x = \omega\partial_y B_z \quad (22)$$

$$E_x = \frac{1}{i(k^2 - \omega^2/c^2)} (k\partial_x E_z + \omega\partial_y B_z) \quad (23)$$

$$= \frac{i}{\omega^2/c^2 - k^2} (k\partial_x E_z + \omega\partial_y B_z) \quad (24)$$

Similarly, we can get the other 3 equations. Multiply 16 by  $k$  and 20 by  $\omega$  and subtract to get

$$E_y = \frac{i}{\omega^2/c^2 - k^2} (k\partial_y E_z - \omega\partial_x B_z) \quad (25)$$

Multiply 16 by  $\omega/c^2$  and 20 by  $k$  and subtract to get

$$B_x = \frac{i}{\omega^2/c^2 - k^2} \left( k\partial_x B_z - \frac{\omega}{c^2} \partial_y E_z \right) \quad (26)$$

Multiply 17 by  $\omega/c^2$  and 19 by  $k$  and add to get

$$B_y = \frac{i}{\omega^2/c^2 - k^2} \left( k\partial_y B_z + \frac{\omega}{c^2} \partial_x E_z \right) \quad (27)$$

To get the wave equations we can apply 7 and 8:

$$\nabla \cdot \mathbf{E} = \frac{i}{\omega^2/c^2 - k^2} [(k\partial_{xx} E_z + \omega\partial_{yx} B_z) + (k\partial_{yy} E_z - \omega\partial_{xy} B_z)] + ikE_z \quad (28)$$

$$= \frac{i}{\omega^2/c^2 - k^2} [k\partial_{xx} E_z + k\partial_{yy} E_z] + ikE_z \quad (29)$$

$$= 0 \quad (30)$$

The wave equation for  $E_z$  is thus

$$[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] E_z = 0 \quad (31)$$

Exactly the same procedure applied to  $\nabla \cdot \mathbf{B} = 0$  gives

$$[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] B_z = 0 \quad (32)$$

Solving these two equations subject to the boundary conditions will give us all 3 components of each field.

#### PINGBACKS

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