For a one dimensional wave (on a string, say) suppose we now place a boundary at the point \( z = 0 \). For a string, this could be a point at which one string is joined with another string of a different mass per unit length. If we send a wave down the string from large negative \( z \), when this wave reaches \( z = 0 \), there will be a reflected wave that returns towards negative \( z \) and a transmitted wave that proceeds beyond \( z = 0 \) towards positive \( z \). We can get some idea of the nature of these reflected and transmitted waves if we impose some boundary conditions at the point \( z = 0 \).

First, we require the wave function \( f \) to be continuous, for the simple reason that there is no break in the string at \( z = 0 \). The second boundary condition requires that the derivative \( \partial f / \partial z \) is also continuous. The reason for this is a bit more subtle. Assuming there is no point mass (such as a knot) at the joining position, if the tangent to the string at that point were not continuous, then the second derivative would be infinite, meaning (from Newton’s law \( F = ma \)) that there would be an infinite force at that point.

To see how the incident, reflected and transmitted waves are related, suppose we have an incident wave \( I(z - v_1 t) \), a reflected wave \( R(z + v_1 t) \) and a transmitted wave \( T(x - v_2 t) \), where \( I \) and \( T \) are moving to the right, with \( I \) defined for \( z < 0 \) and \( T \) for \( z > 0 \), and \( R \) moving to the left for \( z < 0 \).

The continuity of the wave function at \( z = 0 \) gives us

\[
I(-v_1 t) + R(v_1 t) = T(-v_2 t)
\]  

(1)

The continuity of the derivative, if applied directly, just gives the same equation with each function replaced by its derivative, so doesn’t help much:

\[
\frac{\partial I}{\partial z}
\bigg|_{z=0^-} + \frac{\partial R}{\partial z}
\bigg|_{z=0^+} = \frac{\partial T}{\partial z}
\bigg|_{z=0^+}
\]  

(2)

However, if we consider the original definition of a derivative as a limit, we can make some progress. Consider first the derivative of the incident wave just below \( z = 0 \), at time \( t = 0 \):

\[
\frac{\partial I}{\partial z}
\bigg|_{z=0^-} = \lim_{\Delta z \to 0} \frac{I(0) - I(-\Delta z)}{\Delta z}
\]  

(3)
The wave amplitude at the point \((z, t) = (\Delta z, 0)\) will be at \(z = 0\) after it travels the distance \(\Delta z\), which takes a time \(t = \Delta z/v_1\).

By a similar argument, the derivative of the reflected wave is

\[
\frac{\partial R}{\partial z} \bigg|_{z=0^+} = \lim_{\Delta z \to 0} \frac{R(0) - R(-\Delta z)}{\Delta z} \tag{4}
\]

This time, the wave amplitude at \((-\Delta z, 0)\) was at \(z = 0\) at time \(t = -\Delta z/v_1\) since this wave is travelling to the left. Finally, for the transmitted wave

\[
\frac{\partial T}{\partial z} \bigg|_{z=0^-} = \lim_{\Delta z \to 0} \frac{T(\Delta z) - T(0)}{\Delta z} \tag{5}
\]

since this wave is defined for \(z > 0\). The wave amplitude at \((\Delta z, 0)\) was at \(z = 0\) at time \(t = -\Delta z/v_2\) since the transmitted wave is travelling to the right with speed \(v_2\).

We can now use the continuity condition 1 to eliminate either \(R\) or \(T\) from the limits. Start by eliminating \(R\) by evaluating everything at time \(t = -\Delta z/v_1\). The reflected wave amplitude at \(z = 0\) is at time \(t = -\Delta z/v_1\), so we plug this into the argument for \(T\) in 1, so we evaluate \(T\) at \(-v_2 t = v_2 v_1 \Delta z\). Similarly, we plug this time into the argument for \(I\) in 1, so we evaluate \(I\) at \(-v_1 t = \Delta z\). We have:

\[
\lim_{\Delta z \to 0} \frac{R(0) - R(-\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ T(0) - I(0) - \left( T\left( \frac{v_2}{v_1} \Delta z \right) - I(\Delta z) \right) \right] \tag{6}
\]

Now we can insert this into the continuity equation for derivatives 2

\[
\lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ I(0) - I(-\Delta z) + T(0) - I(0) - \left( T\left( \frac{v_2}{v_1} \Delta z \right) - I(\Delta z) \right) \right] = \lim_{\Delta z \to 0} \frac{1}{\Delta z} [T(\Delta z) - T(0)] \tag{7}
\]

\[
\lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ (I(0) - I(-\Delta z)) + (I(\Delta z) - I(0)) \right] = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ \left( T\left( \frac{v_2}{v_1} \Delta z \right) - T(0) \right) + (T(\Delta z) - T(0)) \right] \tag{8}
\]

We can use a Taylor series on the first term on the RHS to get

\[
T\left( \frac{v_2}{v_1} \Delta z \right) = T(0) + \left( \frac{v_2}{v_1} \Delta z \right) \frac{\partial T}{\partial z} + \ldots \tag{9}
\]

Taking the limit to express things in terms of derivatives, we have
\[
\frac{2}{v_1} \frac{\partial I}{\partial z} = \left( \frac{v_2}{v_1} + 1 \right) \frac{\partial T}{\partial z} \tag{10}
\]

This condition is strictly true only at \( z = 0 \), but it must be true for all times. We can convert the derivatives into time derivatives by noting that since \( I = I(z - v_1 t) \) we have

\[
\frac{\partial I}{\partial z} = -\frac{1}{v_1} \frac{\partial I}{\partial t} \tag{11}
\]

Similarly for \( R \) and \( T \):

\[
\frac{\partial T}{\partial z} = -\frac{1}{v_2} \frac{\partial T}{\partial t} \tag{12}
\]

\[
\frac{\partial R}{\partial z} = \frac{1}{v_1} \frac{\partial R}{\partial t} \tag{13}
\]

Returning to (10) we get

\[
-\frac{2}{v_1} \frac{\partial I}{\partial t} = -\frac{1}{v_2} \left( \frac{v_2}{v_1} + 1 \right) \frac{\partial T}{\partial t} \tag{14}
\]

\[
\frac{\partial T}{\partial t} = \frac{2v_2}{v_1 + v_2} \frac{\partial I}{\partial t} \tag{15}
\]

At \( z = 0 \), we can integrate with respect to time to get

\[
T(-v_2 t) = \frac{2v_2}{v_1 + v_2} I(-v_1 t) + K_T \tag{16}
\]

where \( K_T \) is a constant of integration. Although \( I, T \) and \( R \) are functions of both \( z \) and \( t \), they are all actually functions of only one variable, since \( z \) and \( t \) must always occur in the combination \( z \pm v_1 t \). Thus what (16) is saying is that, if we have the incident wave in the form \( I(u) \), then the transmitted wave has the form

\[
T \left( \frac{v_2}{v_1} u \right) = \frac{2v_2}{v_1 + v_2} I \left( \frac{v_1}{v_2} u \right) + K_T \tag{17}
\]

or, conversely

\[
T(u) = \frac{2v_2}{v_1 + v_2} I \left( \frac{v_1}{v_2} u \right) + K_T \tag{18}
\]

For \( T(u) \), \( u = z - v_2 t \) for \( z \geq 0 \) (and all times \( t \)), while for \( I \left( \frac{v_1}{v_2} u \right) \), \( u = z - v_1 t \) for \( z \leq 0 \). If we pick a particular numerical value for \( u \), say 42, then we can write
\[ T(z_T - v_2 t_T) = \frac{2v_2}{v_1 + v_2} I \left( \frac{v_1}{v_2} (z_I - v_1 t_I) \right) + K_T \]  

(19)  

\[ T(42) = \frac{2v_2}{v_1 + v_2} I \left( 42 \frac{v_1}{v_2} \right) + K_T \]  

(20)  

and this equation is valid for all values of \( z \) and \( t \) such that

\[ z_T - v_2 t_T = 42 \quad (z \geq 0) \]  

(21)  

\[ z_I - v_1 t_I = 42 \quad (z \leq 0) \]  

(22)  

That is, the values of \( z_T \) and \( z_I \) need not be equal, and neither must \( t_T = t_I \). All that matters is that \( z_T - v_2 t_T = z_I - v_1 t_I \).

For the reflected wave, we eliminate \( T \) using \( t = -\Delta z/v_2 \):

\[ \lim_{\Delta z \to 0} \frac{T(\Delta z) - T(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ I \left( \frac{v_1}{v_2} \Delta z \right) + R \left( -\frac{v_1}{v_2} \Delta z \right) - I(0) - R(0) \right] \]  

(23)  

Substitute into \( 2 \) and we have

\[ \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ I(0) - I(-\Delta z) + R(0) - R(-\Delta z) \right] = \]  

\[ \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ I \left( \frac{v_1}{v_2} \Delta z \right) + R \left( -\frac{v_1}{v_2} \Delta z \right) - I(0) - R(0) \right] \]  

(24)  

\[ \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ (I(0) - I(-\Delta z)) - \left( I \left( \frac{v_1}{v_2} \Delta z \right) - I(0) \right) \right] = \]  

\[ - \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[ \left( R(0) - R \left( -\frac{v_1}{v_2} \Delta z \right) \right) + (R(0) - R(-\Delta z)) \right] \]  

(25)  

Taking the limit, we have:

\[ \left( 1 - \frac{v_1}{v_2} \right) \frac{\partial I}{\partial z} = - \left( 1 + \frac{v_1}{v_2} \right) \frac{\partial R}{\partial z} \]  

(26)  

\[ - \frac{1}{v_1} \left( 1 - \frac{v_1}{v_2} \right) \frac{\partial I}{\partial t} = - \frac{1}{v_1} \left( 1 + \frac{v_1}{v_2} \right) \frac{\partial R}{\partial t} \]  

(27)  

\[ R(+v_1 t) = \frac{v_2 - v_1}{v_1 + v_2} I(-v_1 t) \]  

(28)  

\[ R(u) = \frac{v_2 - v_1}{v_1 + v_2} I(-u) \]  

(29)
Again, for $R$, $u = z + v_1 t$ with $z \leq 0$ and for $I$, $u = z - v_1 t$ with $z \leq 0$. These results apply to any wave shape, not just to sinusoidal waves.