WAVES - BOUNDARY CONDITIONS

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For a one dimensional wave (on a string, say) suppose we now place a boundary at the point $z=0$. For a string, this could be a point at which one string is joined with another string of a different mass per unit length. If we send a wave down the string from large negative z, when this wave reaches $z=0$, there will be a reflected wave that returns towards negative z and a transmitted wave that proceeds beyond $z=0$ towards positive z. We can get some idea of the nature of these reflected and transmitted waves if we impose some boundary conditions at the point $z=0$.

First, we require the wave function f to be continuous, for the simple reason that there is no break in the string at $z=0$. The second boundary condition requires that the derivative $\partial f/\partial z$ is also continuous. The reason for this is a bit more subtle. Assuming there is no point mass (such as a knot) at the joining position, if the tangent to the string at that point were not continuous, then the second derivative would be infinite, meaning (from Newton's law $F=ma$) that there would be an infinite force at that point.

To see how the incident, reflected and transmitted waves are related, suppose we have an incident wave $I(z-v_1t)$, a reflected wave $R(z+v_1t)$ and a transmitted wave $T(x-v_2t)$, where I and T are moving to the right, with I defined for $z<0$ and T for $z>0$, and R moving to the left for $z<0$.

The continuity of the wave function at $z=0$ gives us

$$I(-v_1t)+R(v_1t)=T(-v_2t) \tag{1}$$

The continuity of the derivative, if applied directly, just gives the same equation with each function replaced by its derivative, so doesn't help much:

$$\left.\frac{\partial I}{\partial z}\right|_{z=0^-}+\left.\frac{\partial R}{\partial z}\right|_{z=0^-}=\left.\frac{\partial T}{\partial z}\right|_{z=0^+} \tag{2}$$

However, if we consider the original definition of a derivative as a limit, we can make some progress. Consider first the derivative of the incident wave just below $z=0$, at time $t=0$:

$$\left.\frac{\partial I}{\partial z}\right|_{z=0^-}=\lim_{\Delta z\to 0}\frac{I(0)-I(-\Delta z)}{\Delta z} \tag{3}$$

The wave amplitude at the point $(z,t) = (-\Delta z, 0)$ will be at $z = 0$ after it travels the distance Δz, which takes a time $t = \Delta z / v_1$.

By a similar argument, the derivative of the reflected wave is

$$\left.\frac{\partial R}{\partial z}\right|_{z=0^-} = \lim_{\Delta z\to 0} \frac{R(0) - R(-\Delta z)}{\Delta z} \tag{4}$$

This time, the wave amplitude at $(-\Delta z, 0)$ was at $z = 0$ at time $t = -\Delta z / v_1$ since this wave is travelling to the left. Finally, for the transmitted wave

$$\left.\frac{\partial T}{\partial z}\right|_{z=0^+} = \lim_{\Delta z\to 0} \frac{T(\Delta z) - T(0)}{\Delta z} \tag{5}$$

since this wave is defined for $z > 0$. The wave amplitude at $(\Delta z, 0)$ was at $z = 0$ at time $t = -\Delta z / v_2$ since the transmitted wave is travelling to the right with speed v_2.

We can now use the continuity condition 1 to eliminate either R or T from the limits. Start by eliminating R by evaluating everything at time $t = -\Delta z / v_1$. The reflected wave amplitude at $z = 0$ is at time $t = -\Delta z / v_1$, so we plug this into the argument for T in 1, so we evaluate T at $-v_2 t = \frac{v_2}{v_1}\Delta z$. Similarly, we plug this time into the argument for I in 1, so we evaluate I at $-v_1 t = \Delta z$. We have:

$$\lim_{\Delta z\to 0} \frac{R(0) - R(-\Delta z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[T(0) - I(0) - \left(T\left(\frac{v_2}{v_1}\Delta z\right) - I(\Delta z)\right)\right] \tag{6}$$

Now we can insert this into the continuity equation for derivatives 2.

$$\lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[I(0) - I(-\Delta z) + T(0) - I(0) - \left(T\left(\frac{v_2}{v_1}\Delta z\right) - I(\Delta z)\right)\right] = \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[T(\Delta z) - T(0)\right] \tag{7}$$

$$\lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[(I(0) - I(-\Delta z)) + (I(\Delta z) - I(0))\right] = \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\left(T\left(\frac{v_2}{v_1}\Delta z\right) - T(0)\right) + (T(\Delta z) - T(0))\right] \tag{8}$$

We can use a Taylor series on the first term on the RHS to get

$$T\left(\frac{v_2}{v_1}\Delta z\right) = T(0) + \left(\frac{v_2}{v_1}\Delta z\right)\frac{\partial T}{\partial z} + \ldots \tag{9}$$

Taking the limit to express things in terms of derivatives, we have

$$2\frac{\partial I}{\partial z} = \left(\frac{v_2}{v_1}+1\right)\frac{\partial T}{\partial z} \tag{10}$$

This condition is strictly true only at $z=0$, but it must be true for all times. We can convert the derivatives into time derivatives by noting that since $I=I(z-v_1t)$ we have

$$\frac{\partial I}{\partial z}=-\frac{1}{v_1}\frac{\partial I}{\partial t} \tag{11}$$

Similarly for R and T:

$$\frac{\partial T}{\partial z}=-\frac{1}{v_2}\frac{\partial T}{\partial t} \tag{12}$$

$$\frac{\partial R}{\partial z}=\frac{1}{v_1}\frac{\partial R}{\partial t} \tag{13}$$

Returning to 10 we get

$$-\frac{2}{v_1}\frac{\partial I}{\partial t}=-\frac{1}{v_2}\left(\frac{v_2}{v_1}+1\right)\frac{\partial T}{\partial t} \tag{14}$$

$$\frac{\partial T}{\partial t}=\frac{2v_2}{v_1+v_2}\frac{\partial I}{\partial t} \tag{15}$$

At $z=0$, we can integrate with respect to time to get

$$T(-v_2t)=\frac{2v_2}{v_1+v_2}I(-v_1t)+K_T \tag{16}$$

where K_T is a constant of integration. Although I, T and R are functions of both z and t, they are all actually functions of only one variable, since z and t must always occur in the combination $z\pm v_{1,2}t$. Thus what 16 is saying is that, if we have the incident wave in the form $I(u)$, then the transmitted wave has the form

$$T\left(\frac{v_2}{v_1}u\right)=\frac{2v_2}{v_1+v_2}I(u)+K_T \tag{17}$$

or, conversely

$$T(u)=\frac{2v_2}{v_1+v_2}I\left(\frac{v_1}{v_2}u\right)+K_T \tag{18}$$

For $T(u)$, $u=z-v_2t$ for $z\ge 0$ (and all times t), while for $I\left(\frac{v_1}{v_2}u\right)$, $u=z-v_1t$ for $z\le 0$. If we pick a particular numerical value for u, say 42, then we can write

$$T(z_T - v_2 t_T) = \frac{2v_2}{v_1+v_2} I\left(\frac{v_1}{v_2}(z_I - v_1 t_I)\right) + K_T \tag{19}$$

$$T(42) = \frac{2v_2}{v_1+v_2} I\left(42\frac{v_1}{v_2}\right) + K_T \tag{20}$$

and this equation is valid for *all* values of z and t such that

$$z_T - v_2 t_T = 42 \ (z \geq 0) \tag{21}$$

$$z_I - v_1 t_I = 42 \ (z \leq 0) \tag{22}$$

That is, the values of z_T and z_I need not be equal, and neither must $t_T = t_I$. All that matters is that $z_T - v_2 t_T = z_I - v_1 t_I$.

For the reflected wave, we eliminate T using 1 at time $t = -\Delta z / v_2$:

$$\lim_{\Delta z \to 0} \frac{T(\Delta z) - T(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[I\left(\frac{v_1}{v_2}\Delta z\right) + R\left(-\frac{v_1}{v_2}\Delta z\right) - I(0) - R(0)\right] \tag{23}$$

Substitute into 2 and we have

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[I(0) - I(-\Delta z) + R(0) - R(-\Delta z)\right] = \\ \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[I\left(\frac{v_1}{v_2}\Delta z\right) + R\left(-\frac{v_1}{v_2}\Delta z\right) - I(0) - R(0)\right] \end{aligned} \tag{24}$$

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[(I(0) - I(-\Delta z)) - \left(I\left(\frac{v_1}{v_2}\Delta z\right) - I(0)\right)\right] = \\ -\lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\left(R(0) - R\left(-\frac{v_1}{v_2}\Delta z\right)\right) + (R(0) - R(-\Delta z))\right] \end{aligned} \tag{25}$$

Taking the limit, we have:

$$\left(1 - \frac{v_1}{v_2}\right)\frac{\partial I}{\partial z} = -\left(1 + \frac{v_1}{v_2}\right)\frac{\partial R}{\partial z} \tag{26}$$

$$-\frac{1}{v_1}\left(1 - \frac{v_1}{v_2}\right)\frac{\partial I}{\partial t} = -\frac{1}{v_1}\left(1 + \frac{v_1}{v_2}\right)\frac{\partial R}{\partial t} \tag{27}$$

$$R(+v_1 t) = \frac{v_2 - v_1}{v_1 + v_2} I(-v_1 t) \tag{28}$$

$$R(u) = \frac{v_2 - v_1}{v_1 + v_2} I(-u) \tag{29}$$

Again, for R, $u = z + v_1 t$ with $z \leq 0$ and for I, $u = z - v_1 t$ with $z \leq 0$. These results apply to any wave shape, not just to sinusoidal waves.