CLASSICAL FIELD THEORY

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I'll summarize here Coleman's description of classical field theory in his Chapter 4 as it contains several concepts that are fundamental to the physics of fields theories (both classical and quantum) that do not seem to be emphasized in many books. This post is concerned with *classical* field theory.

The first point is that there is a conceptual shift from classical particle theory to field theory. In particle theory, the components of a physical system are individual particles, which are described by their generalized coordinates $q^a(t)$. Each coordinate is a function of time t, with the equations of motion (given by the Euler-Lagrange equations) used to describe each coordinate as a function of time.

In classical field theory, the physical system consists of one or more fields $\phi^a(\boldsymbol{x},t)$ that are assumed to exist over all space and time (or possibly some restricted region thereof). The equations of motion (also obtained from Euler-Lagrange equations) now describe values of the fields at each location in space as functions of time. The crucial point is that the spatial position $\boldsymbol{x}$ is now considered as a label for the field rather than as a coordinate that evolves in time. In continuous space, we therefore have an infinite number of fields, with one field ϕ^a at each point in space. In effect, the label a on the particle theory's coordinate q^a has now become a compound label $(a,\boldsymbol{x})$ with the a labelling the type of field and the $\boldsymbol{x}$ labelling the point in space where that field is evaluated.

The most common classical fields are the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ described by Maxwell's equations. A solution of Maxwell's equations for a particular configuration gives us values of $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$ as functions of both space and time.

Classical field theory uses a Lagrangian in analogy to the Lagrangian of classical particle mechanics. Since our ultimate goal is to obtain a theory that is consistent with special relativity, we want a theory that is Lorentz invariant. In relativity, space and time appear on an equal footing, so the usual assumption is that they must be treated equally in the Lagrangian. In particle mechanics, we restricted the Lagrangian to be a function of the generalized coordinates q^a and their first time derivatives $\dot{q}^a$ (and possibly explicitly of the time t). In a Lorentz invariant field theory, therefore, we consider Lagrangians that are functions (technically, functionals) of the fields $\phi^a(\boldsymbol{x},t)$ and their first derivatives with respect to the three spatial coordinates and time $\partial_\mu \phi^a(\boldsymbol{x},t)$ (and possibly spacetime x as well).

The total Lagrangian is now considered to be the space integral of the *Lagrangian density* $\mathscr{L}$:

$$L = \int d^3\boldsymbol{x}\, \mathscr{L}\left(\phi^a(x), \partial_\mu \phi^a(x), x\right) \tag{1}$$

Here a bold $\boldsymbol{x}$ indicates the 3-vector of spatial coordinates and x indicates a 4-vector of spacetime. The Greek index μ runs from 0 to 3, with 0 being time.

The action, as with particle mechanics, is the integral of the total Lagrangian L over some time interval:

$$\begin{aligned} \mathcal{S} &= \int_{t_1}^{t_2} dt\, L & (2)\\ &= \int d^4x\, \mathscr{L}\left(\phi^a(x), \partial_\mu \phi^a(x), x\right) & (3) \end{aligned}$$

where in the second line, the 4-dimensional integral is taken over all space, but only over the time interval $[t_1, t_2]$.

From here, the requirement that the variation of $\mathcal{S}$ in the interval $[t_1, t_2]$ vanishes is used to derive the Euler-Lagrange equations for classical field theory. Coleman gives the details, with the result

$$\boxed{\frac{\partial \mathscr{L}}{\partial \phi^a} - \partial_\mu \pi_a^\mu = 0} \tag{4}$$

where

$$\pi_a^\mu = \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi^a\right)} \tag{5}$$

We can derive a Hamiltonian form for classical field theory by following steps similar to that for particle theory. Coleman starts with the observation that in particle theory, an infinitestimal change in the Lagrangian is given by

$$dL = p_a d\dot{q}^a + \ldots \tag{6}$$

where the dots indicate terms not involving time derivatives. For field theory, we start with 1 and take its differential:

$$\delta L = \delta \int d^3\boldsymbol{x}\, \mathscr{L} \tag{7}$$

$$= \int d^3\boldsymbol{x}\, \frac{\partial \mathscr{L}}{\partial\left(\partial_0 \phi^a\right)} \delta \dot{\phi}^a + \ldots \tag{8}$$

$$= \int d^3\boldsymbol{x}\, \pi_a^0 \delta \dot{\phi}^a + \ldots \tag{9}$$

Again, the dots indicate terms not involving time derivatives.

Since p_a is the momentum in particle mechanics, we identify $\pi_a \equiv \pi_a^0$ as a canonical momentum density in field theory. Note that it is only the 0 component of π_a^μ in 5 that is analogous to the particle theory momentum, so it's not correct to think of π_a^μ as some 4-d generalization of the momentum p_a.

In particle mechanics, the Hamiltonian is defined by the Legendre transformation

$$H = p_a \dot{q}^a - L \tag{10}$$

so the analogous equation in field theory is

$$H = \int d^3\boldsymbol{x} \left(\pi_a \dot{\phi}^a - \mathscr{L}\right) \tag{11}$$

The integrand is the *Hamiltonian density*

$$\mathscr{H} \equiv \pi_a \dot{\phi}^a - \mathscr{L} \tag{12}$$

Coleman does an example with the Lagrangian

$$\mathscr{L} = \pm \frac{1}{2}\left(a \partial_\mu \phi \partial^\mu \phi + b \phi^2\right) \tag{13}$$

where a and b are constants. In this case, there is only a single field so the index a on the field ϕ^a has been dropped. The a in this equation is just a constant; it's not the same as the index a on ϕ^a.

He shows that applying the above theory to this Lagrangian (technically, Lagrangian density, but usually it's just called the Lagrangian) and requiring that the energy be positive results in

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - \mu^2 \phi^2\right) \tag{14}$$

$$\mathscr{H} = \frac{1}{2}\left(\pi^2 + |\nabla \phi|^2 + \mu^2 \phi^2\right) \tag{15}$$

where

$$\mu^2 = -\frac{b}{a} \tag{16}$$

is a *positive* constant and

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \tag{17}$$

The equation of motion obtained from 4 is the Klein-Gordon equation

$$\Box^2 \phi + \mu^2 \phi = 0 \tag{18}$$

where the d'Alembertian operator is

$$\Box^2 \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{19}$$