CONNECTED AND DISCONNECTED WICK DIAGRAMS

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Each term in a Wick expansion of a time-ordered product of fields can be represented as a Wick diagram. A Wick diagram is essentially a graph (in the mathematical sense of an object consisting of vertices and edges connecting them). A graph is called *connected* if there is a path along the edges from any given vertex to any other vertex in the graph. A *disconnected* graph is, not surprisingly, a graph that is not connected. That is, there are two or more portions of the graph that cannot be linked to all other vertices in the graph by following a path along the edges.

The Wick expansion is given by

$$T(\phi_1\phi_2\ldots\phi_n) = :\phi_1\phi_2\ldots\phi_n: \tag{1}$$

$$+:\overbrace{\phi_1\phi_2}\ldots\phi_n: + (\text{all other terms with 1 contraction}) \tag{2}$$

$$+:\overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4}\ldots\phi_n: + (\text{all other 2 contraction terms}) \tag{3}$$

$$+\ldots+\left(\text{all terms with }\frac{1}{2}n\text{ or }\frac{1}{2}(n-1)\text{ contractions}\right) \tag{4}$$

This expansion is for a given order in the expansion of the original Dyson exponential

$$U_I(t,t') = T\exp\left(-i\int_{t'}^{t}dt''H_I(t'')\right) \tag{5}$$

The nth term in the expansion has the form

$$\frac{(-i)^n}{n!}T\int_1\int_2\cdots\int_n d^4x_1d^4x_2\ldots d^4x_n\mathscr{H}_{I1}\mathscr{H}_{I2}\ldots\mathscr{H}_{In} \tag{6}$$

where $\mathscr{H}_{Ii} = \mathscr{H}_I(x_i)$.

As we go to ever higher orders in the expansion, the number of field terms in the time-ordered product increases and thus the number of terms in

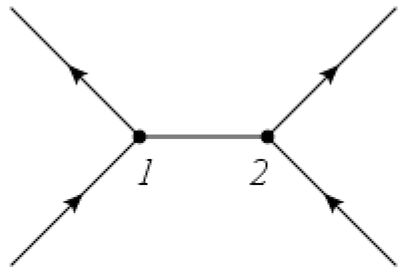

FIGURE 1. The diagram corresponding to $:\psi_1^*\psi_1\overbracket{\phi_1\psi_2^*\psi_2\phi_2}:$

the corresponding Wick expansion also increases. Since there is one vertex in a Wick diagram for each spacetime point x_i in the integral, the Wick diagrams for the nth order in the expansion all have n vertices. The number of edges depends on the number of field operators in the Hamiltonian, and thus varies from one model to the next.

A given Wick diagram from the nth order term may be either connected or disconnected. A disconnected diagram is composed of a number of smaller, connected diagrams. Suppose we have a connected sub-diagram with r vertices, where $r < n$. Then this diagram is one of the Wick diagrams from the rth order term in the overall expansion.

If we plug the Wick expansion 2 into the nth order integral 6 we get a sum of integrals over normal-ordered fields. For example, with the Hamiltonian used earlier

$$\mathscr{H}_I = gf(t)\,\psi^*\psi\phi \tag{7}$$

one of the terms in the second order expansion gives the integral

$$\frac{(-i)^n}{n!}\int_1\int_2 d^4x_1 d^4x_2 \psi_1^*\psi_1\overbracket{\phi_1\psi_2^*\psi_2\phi_2} \tag{8}$$

This corresponds to the diagram in Fig. 1, which is a connected diagram.

For any diagram D, we can write the corresponding term in the expansion in the form

$$\frac{:O(D):}{n(D)!} \tag{9}$$

For example, the term 8 can be written as

$$\frac{O(D)}{n(D)!} = \frac{(-i)^n}{n(D)!}\int_1\int_2 d^4x_1 d^4x_2 \psi_1^*\psi_1\overbracket{\phi_1\psi_2^*\psi_2\phi_2} \tag{10}$$

where

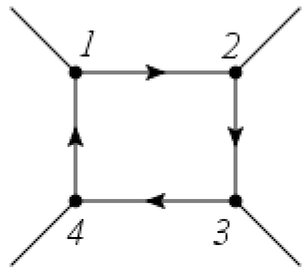

FIGURE 2. The vertices in this diagram can be rotated through 90 degrees without changing the physics.

$$O(D) = (-i)^n \int_1 \int_2 d^4x_1 d^4x_2 \psi_1^* \psi_1 \overline{\phi_1 \psi_2^* \psi_2 \phi_2} \tag{11}$$

Note that the factorial $n(D)!$ is the number of ways of permuting the n vertices in the diagram, and that it is also the factorial term that arises from expanding the original exponential 5. Coleman writes it as a separate factor in order to cancel it later.

At this point, we must note that, although there are $n(D)!$ ways of rearranging the vertices in a Wick diagram, this doesn't mean that all the diagrams you get by doing this are physically different. For example, in Fig. 2, if we permute the vertices according to $1 \to 2 \to 3 \to 4 \to 1$, we are merely rotating the diagram by 90 degrees without changing anything, so all four of these permutations are physically equivalent.

For a diagram D, Coleman defines the number $S(D)$ to be the number of physically equivalent diagrams that can be obtained by permuting the vertices. Note that $S(D) \leq n(D)!$. The total number of permutations $n(D)!$ is therefore the number of physically distinct arrangements of the vertices multiplied by the factor $S(D)$. For example, in Fig. 2, $S(D) = 4$, since for any given permutation of the vertices, there are 3 other permutations that are physically equivalent. Thus the number of distinct permutations multiplied by $S(D)$ must equal the total number of permutations, which is $4! = 24$. So the number of distinct diagrams is $\frac{4!}{4} = 6$. In general, the number of distinct patterns is $\frac{n(D)!}{S(D)}$

Returning to 9, the total contribution from all the diagrams in a given pattern is therefore the quantity 9 multiplied by the number of physically distinct patterns, so we have

$$\begin{pmatrix}\text{sum of all diagrams} \\ \text{in a given pattern}\end{pmatrix} = \frac{n(D)!}{S(D)} \frac{:O(D):}{n(D)!} = \frac{:O(D):}{S(D)} \tag{12}$$

Although the number of Wick diagrams for a given order in the expansion of 5 is finite (as you can see by noting that there are a finite number of terms in 2), the total number of Wick diagrams for a given hamiltonian will, in general, be infinite, since there could be an infinite number of terms in the expansion of the exponential. In general, diagrams from higher orders can be connected or disconnected, but the connected components of disconnected diagrams are themselves diagrams from lower orders in the expansion. Coleman proves the elegant theorem

$$\sum(\text{all Wick diagrams}) = :\exp\left(\sum\text{connected Wick diagrams}\right): \qquad (13)$$

Coleman's proof is, I believe, quite clear so I won't reproduce it here. I'll just summarize his notation for reference.

We suppose that we have a complete set $D_r^{(c)}$ of connected diagrams. The (c) means 'connected' and the r indicates the specific pattern (that is, a physically distinct pattern, and not an equivalent pattern obtained by permutation, as described above), so $r = 1, 2, \ldots, \infty$. Note that r does *not* represent the order in the expansion; it's merely a label for each possible connected diagram for the hamiltonian under consideration. It extends to infinity, since the number of terms in the expansion of the exponential can be infinite, and each of these terms could have one or more connected diagrams.

A general diagram (connected or disconnected) D is composed of one or more of these connected diagrams, so let n_r be the number of times the diagram $D_r^{(c)}$ appears in the diagram D. If D is connected, then $n_r = 1$ for exactly one of the diagrams in $D_r^{(c)}$ and $n_r = 0$ for all the others. If D is disconnected, then either $n_r \neq 0$ for two or more diagrams in $D_r^{(c)}$, or $n_r > 1$ for one diagram in $D_r^{(c)}$, or both.

For a disconnected diagram $D^{(d)}$, its corresponding $O\left(D^{(d)}\right)$ (as in 9) is the product of the corresponding quantities for each connected sub-diagram. That is

$$:O^{(d)}: = :\Pi_{r=1}^{\infty}\left[O\left(D_r^{(c)}\right)\right]^{n_r}: \qquad (14)$$

From here, the proof of 13 is fairly straightforward and is given by Coleman's eqns 8.51 to 8.56. The final result is

$$\boxed{U_I(\infty, -\infty) = :\exp\sum_{r=1}^{\infty}\left(\frac{O\left(D_r^{(c)}\right)}{S\left(D_r^{(c)}\right)}\right):} \qquad (15)$$

It's worth stating again that the various O terms here are all *operators*, as they are all integrals similar to 8. Each of the field operators in this integral consists of creation and annihilation operators. Each of the Os can be represented by a Wick diagram. Thus the evolution operator $U_I(\infty,-\infty)$ is the exponential of a sum of operators, with each operator representing a distinct, connected Wick diagram.

The catch is that we need to know the number $S\left(D_r^{(c)}\right)$ for each connected diagram. That is, we need to know how many permutations of the indices in each diagram are physically equivalent. This can be tricky to work out (see Coleman's Fig. 8.10 for an example with 6 vertices).

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