COUNTERTERMS IN INTERACTION HAMILTONIANS

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In the previous post, we looked at the exact solution of Coleman's Model 1 interaction hamiltonian

$$\mathscr{H}_I = g\rho(x)\phi(x) \tag{1}$$

In his Model 2, the interaction hamiltonian is now

$$\mathscr{H}_I = g\phi(x)\rho(\boldsymbol{x}) \tag{2}$$

The difference is that, in model 1 the function $\rho(x)$ depended on all of spacetime, and was designed so that it went to zero for times in the distant past or distant future. In Model 2, the function $\rho(\boldsymbol{x})$ depends on space only, and is independent of time. The consequence of this is that the interaction is constant over all time. To cope with this, Coleman introduces an artificial 'adiabatic function' $f(t,T,\Delta)$ which is shown in his Fig. 9.1. Its essential features are that it is zero for times in the distant past, then slowly switches on over the time interval $-\frac{T}{2}-\Delta$ to $-\frac{T}{2}$. It then is constant at $f=1$ for times from $-\frac{T}{2}$ to $+\frac{T}{2}$, after which it decreases back to zero over the time interval $+\frac{T}{2}$ to $\frac{T}{2}+\Delta$.

The purpose of f is to allow us to treat Model 2 so that in the distant past and distant future it is a free field, in the same way as we treated Model 1. The catch here is that the adiabatic function f is purely artificial, and thus the true behaviour of Model 2 should not depend on T or Δ in the limit $T\to\infty$.

In eqn 9.3, Coleman states that matrix element of the S matrix should satisfy

$$\lim_{T\to\infty}\langle 0|U_I(T,-T)|0\rangle = \langle 0|\mathrm{S}|0\rangle = 1 \tag{3}$$

I think the rationale for this is that we can use the expression for the S matrix in terms of connected Wick diagrams.

$$S = :\exp \sum_{r=1}^{\infty} \left(\frac{O\left(D_r^{(c)}\right)}{S\left(D_r^{(c)}\right)} \right): \tag{4}$$

The expansion of the exponential gives rise to an initial 1 followed by the results of the various diagrams. Any diagram that is fully contracted, that is, that has no external lines, gives rise to a pure number when evaluated. Any diagram that *does* have one or more external lines will give a zero contribution to 3, since such a diagram contains both annihilation operators (which give zero when acting on $|0\rangle$) and creation operators (which give zero when operating on the bra term $\langle 0|$). Thus only fully contracted terms contribute to 3 (in addition to the initial 1 in the expansion of the exponential). These contracted terms can result in 3 differing from 1, and Coleman shows that in this theory where we have an auxiliary adiabatic function $f(t)$ which slowly switches the interaction on and then off again, these extra terms give rise to inconsistent results.

In section 9.1, Coleman goes through a calculation of $\langle 0|\mathrm{S}|0\rangle$ by starting with the free vacuum state $|0\rangle$ which is valid up to $t = -\frac{T}{2} - \Delta$. As the adiabatic function $f(t)$ slowly switches the source on, we can use the adiabatic theorem to state that the ground state evolves from the free vacuum state to the ground state of the full interacting system, given by $|0\rangle_P$. The ground state energy of the free vacuum is 0, but the ground state energy of the interacting state may not be zero. Coleman rather confusingly calls this interacting state energy E_0, so don't confuse it with the zero ground state energy of the free hamiltonian (which is usually written as H_0).

By applying the adiabatic theorem over the time intervals where f is slowly switched on, and then slowly switched off again, Coleman shows in eqns 9.4 through 9.11 that, using this reasoning

$$\langle 0|\mathrm{S}|0\rangle = e^{-i(\gamma_+ + \gamma_- + E_0 T)} \tag{5}$$

which disagrees with 3. In particular, the vacuum-to-vacuum matrix element depends on the time T which we don't want, because we ultimately want to eliminate T from the calculation.

Although Coleman doesn't mention this, my suspicion is that this non-zero ground state energy E_0 is the result of the fully contracted terms in the expansion 4. In other words, we'd like to fix (or, to use the technical term, *renormalize*) the model so that the ground state of the interacting system remains at zero. I'm not entirely certain this is the correct interpretation, but please leave a comment if you agree or disagree.

Coleman solves this problem by introducing a *counterterm* into H_I (which is the spatial integral of the hamiltonian density 2). The original overall hamiltonian (including the adiabatic function f) is the space integral of 2:

$$H_I = gf(t)\int d^3\boldsymbol{x}\phi(x)\rho(\boldsymbol{x}) \tag{6}$$

This counterterm is a number called a, and is inserted outside the spatial integral, so we have

$$H_I = \left[gf(t)\int d^3\boldsymbol{x}\phi(x)\rho(\boldsymbol{x})\right] - af(t) \tag{7}$$

The idea is that, when we integrate this over time (as we did to get the erroneous S-matrix 5), the contribution to the phase from a will cancel the phase in 5. Since $f(t,T,\Delta)$ is 1 for a time interval T and less than 1 for the two tails, each of width Δ, we have

$$a\int dt\ f(t) = a(T+\mathcal{O}(\Delta)) \tag{8}$$

We would therefore like this to cancel the phase in 5, so we'd like, in the limit $T\to\infty$:

$$a(T+\mathcal{O}(\Delta)) = \gamma_+ + \gamma_- + E_0 T \tag{9}$$

By factoring out T, we get

$$\lim_{T\to\infty} aT\left(1+\mathcal{O}\left(\frac{\Delta}{T}\right)\right) = \lim_{T\to\infty}(\gamma_+ + \gamma_- + E_0 T) \tag{10}$$

The phases γ_+ and γ_- are finite phases obtained from applying the adiabatic theorem to the two tails of the f function, so in the limit $T\to\infty$, they disappear from the RHS, and we end up with

$$a = E_0 \tag{11}$$

That is, the counterterm is the ground state energy in the interacting system. If my interpretation above is correct, this result isn't surprising, since we are just subtracting the interaction ground state energy E_0 from the interaction hamiltonian in order to reduce the ground state energy back to zero. Cancelling the extra phase with the counterterm thus results in the desired result 3 for the vacuum-to-vacuum S matrix element.

Using arguments similar to those in the earlier model, Coleman shows that the only component in the expansion of 4 that contains operators is

$$-ig\int\frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}}\left(\tilde{f}(\omega_{\boldsymbol{p}})\tilde{\rho}(\boldsymbol{p})a_{\boldsymbol{p}}+\tilde{f}(\omega_{\boldsymbol{p}})^{*}\tilde{\rho}(\boldsymbol{p})^{*}a_{\boldsymbol{p}}^{\dagger}\right) \tag{12}$$

where the tilde indicates a Fourier transform. In the limit $T\to\infty$, $f(t)$ becomes a constant with value 1, so its Fourier transform is a delta function that peaks at $\omega_{\boldsymbol{p}}=0$. This means that at any energy $\omega_{\boldsymbol{p}}$ of physical relevance, the value of $\tilde{f}(\omega_{\boldsymbol{p}})$ is effectively zero, so this term contributes nothing to 3.

Since $\rho(\boldsymbol{x})$ does not depend on time, it is constant over all time and is therefore a static source, so no mesons are either absorbed or emitted by this source. Remember that, in Model 1, the probability of there being n mesons in the final state is

$$P(n)=e^{-\alpha}\frac{\alpha^{n}}{n!} \tag{13}$$

where

$$\alpha=\int d^3\boldsymbol{p}\,|h(\boldsymbol{p})|^2 \tag{14}$$

and

$$h(\boldsymbol{p})=\frac{-ig\tilde{\rho}(\boldsymbol{p},\omega_{\boldsymbol{p}})}{(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}} \tag{15}$$

Here, $\tilde{\rho}$ is the Fourier transform of the Model 1 source function $\rho(x)$, which can be non-zero on the mass shell, where $\omega_{\boldsymbol{p}}^2=\boldsymbol{p}^2+\mu^2$. In Model 2, since $\rho(\boldsymbol{x})$ is constant in time, its Fourier transform in the time dimension is the Fourier transform of a constant, which is a delta function peaked at $\omega_{\boldsymbol{p}}=0$. Thus *this* Fourier transform is zero on the mass shell, and thus the probability of there being $n>0$ mesons in the final state is zero. Another way of saying this is that the matrix element of S between initial and final vacuum states should be 1.

PINGBACKS

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