CREATION AND ANNIHILATION OPERATORS - NORMALIZATION

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The number operators are defined in terms of the creation and annihilation operators for the free scalar Hamiltonian as

$$N_a(\mathbf{k}) = a^\dagger(\mathbf{k})\, a(\mathbf{k}) \tag{1}$$

$$N_b(\mathbf{k}) = b^\dagger(\mathbf{k})\, b(\mathbf{k}) \tag{2}$$

We've seen that $a^\dagger(\mathbf{k})$ creates a particle of energy $\omega_{\mathbf{k}}$ when it operates on a state, and $a(\mathbf{k})$ destroys a particle with energy $\omega_{\mathbf{k}}$ when it operates on a state (if that state contains such a particle). That is, the state $a^\dagger(\mathbf{k})\,|n_{\mathbf{k}}\rangle$ is an eigenstate of $N_a(\mathbf{k})$ with eigenvalue $n_{\mathbf{k}}+1$ and $a(\mathbf{k})\,|n_{\mathbf{k}}\rangle$ is an eigenstate of $N_a(\mathbf{k})$ with eigenvalue $n_{\mathbf{k}}-1$. However, if we require all multiparticle states to be normalized, so that $\langle n_{\mathbf{k}}\,|n_{\mathbf{k}}\rangle = 1$, the states $a^\dagger(\mathbf{k})\,|n_{\mathbf{k}}\rangle$ and $a(\mathbf{k})\,|n_{\mathbf{k}}\rangle$ do not produce normalized states. Rather, we have

$$a^\dagger(\mathbf{k})\,|n_{\mathbf{k}}\rangle = A\,|n_{\mathbf{k}}+1\rangle \tag{3}$$

$$a(\mathbf{k})\,|n_{\mathbf{k}}\rangle = B\,|n_{\mathbf{k}}-1\rangle \tag{4}$$

for some constants A and B that are determined by requiring normalization.

To find A and B, we can take the modulus of the states above. We get (we'll leave off the $(\mathbf{k})$ dependence of the $a^\dagger$ and a operators to save typing; everything in what follows occurs at wave number $\mathbf{k}$; we'll also assume A and B are real, although they could also be multiplied by some phase factor $e^{i\alpha}$, but this just complicates things unnecessarily). By using the commutation relation

$$\left[a, a^\dagger\right] = 1 \tag{5}$$

we get, from 3

$$\left\langle n_{\mathbf{k}}\left|aa^{\dagger}\right|n_{\mathbf{k}}\right\rangle = A^{2}\left\langle n_{\mathbf{k}}+1\left|n_{\mathbf{k}}+1\right.\right\rangle = A^{2} \tag{6}$$

$$\left\langle n_{\mathbf{k}}\left|aa^{\dagger}\right|n_{\mathbf{k}}\right\rangle = \left\langle n_{\mathbf{k}}\left|a^{\dagger}a+1\right|n_{\mathbf{k}}\right\rangle \tag{7}$$

$$= \left\langle n_{\mathbf{k}}\left|N_{a}+1\right|n_{\mathbf{k}}\right\rangle \tag{8}$$

$$= \left(n_{\mathbf{k}}+1\right)\left\langle n_{\mathbf{k}}\left|n_{\mathbf{k}}\right.\right\rangle \tag{9}$$

$$= \left(n_{\mathbf{k}}+1\right) \tag{10}$$

$$A = \sqrt{n_{\mathbf{k}}+1} \tag{11}$$

Therefore

$$a^{\dagger}\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle = \sqrt{n_{\mathbf{k}}+1}\left|n_{\mathbf{k}}+1\right\rangle \tag{12}$$

For the annihilation operator, we have from 4:

$$\left\langle n_{\mathbf{k}}\left|a^{\dagger}a\right|n_{\mathbf{k}}\right\rangle = B^{2}\left\langle n_{\mathbf{k}}-1\left|n_{\mathbf{k}}-1\right.\right\rangle = B^{2} \tag{13}$$

$$\left\langle n_{\mathbf{k}}\left|a^{\dagger}a\right|n_{\mathbf{k}}\right\rangle = \left\langle n_{\mathbf{k}}\left|N_{a}\right|n_{\mathbf{k}}\right\rangle \tag{14}$$

$$= n_{\mathbf{k}}\left\langle n_{\mathbf{k}}\left|n_{\mathbf{k}}\right.\right\rangle \tag{15}$$

$$= n_{\mathbf{k}} \tag{16}$$

$$B = \sqrt{n_{\mathbf{k}}} \tag{17}$$

Therefore

$$a\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle = \sqrt{n_{\mathbf{k}}}\left|n_{\mathbf{k}}-1\right\rangle \tag{18}$$

This relation implies that applying $a\left(\mathbf{k}\right)$ to a state that contains no particles with energy $\omega_{\mathbf{k}}$ (that is, where $n_{\mathbf{k}}=0$) produces 0. In particular, if we apply $a\left(\mathbf{k}\right)$ to the vacuum state, we end up with no state at all:

$$a\left(\mathbf{k}\right)\left|0\right\rangle = 0 \tag{19}$$

Note that $\left|0\right\rangle$ and 0 aren't the same thing: $\left|0\right\rangle$ is a quantum state with no particles in it, while 0 is mathematically zero, that is, nothing. As an analogy, $\left|0\right\rangle$ is like having a bucket with nothing in it, while 0 corresponds to removing the bucket as well.

We can repeat exactly the same calculations for the antiparticle operators $b^{\dagger}$ and b and get the results

$$b^{\dagger}\left(\mathbf{k}\right)\left|\bar{n}_{\mathbf{k}}\right\rangle = \sqrt{\bar{n}_{\mathbf{k}}+1}\left|\bar{n}_{\mathbf{k}}+1\right\rangle \tag{20}$$

$$b\left(\mathbf{k}\right)\left|\bar{n}_{\mathbf{k}}\right\rangle = \sqrt{\bar{n}_{\mathbf{k}}}\left|\bar{n}_{\mathbf{k}}-1\right\rangle \tag{21}$$

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