CREATION AND DESTRUCTION OPERATORS FOR A FREE SCALAR FIELD

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Post date: 20 August 2023.

The scalar, free-field Hamiltonian derived earlier is

$$H=\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left[N_a(\mathbf{k})+\frac{1}{2}+N_b(\mathbf{k})+\frac{1}{2}\right] \tag{1}$$

where the operators are

$$N_a(\mathbf{k})=a^{\dagger}(\mathbf{k})\,a(\mathbf{k}) \tag{2}$$

$$N_b(\mathbf{k})=b^{\dagger}(\mathbf{k})\,b(\mathbf{k}) \tag{3}$$

These operators can be interpreted as number operators. That is, when they operate on a state $|n_{\mathbf{k}}\rangle$ containing $n_{\mathbf{k}}$ particles with energy $\omega_{\mathbf{k}}$, their eigenvalues are just $n_{\mathbf{k}}$:

$$N_a(\mathbf{k})\,|n_{\mathbf{k}}\rangle=n_{\mathbf{k}}\,|n_{\mathbf{k}}\rangle \tag{4}$$

With this interpretation, and the commutation relations for the a and b operators, we can derive the fact that $a^{\dagger}(\mathbf{k})$ is a *creation operator* for type a particles with energy $\omega_{\mathbf{k}}$, $a(\mathbf{k})$ is a *destruction operator* for the same type of particle, and $b^{\dagger}(\mathbf{k})$ and $b(\mathbf{k})$ are the analogous operators for type b particles (antiparticles).

Suppose we operate on the state $|n_{\mathbf{k}}\rangle$ with $a^{\dagger}(\mathbf{k})$. To see that this creates a particle with energy $\omega_{\mathbf{k}}$, we operate on the resulting state with $N_a(\mathbf{k})$:

$$\begin{aligned}
N_a(\mathbf{k})\,a^{\dagger}(\mathbf{k})\,|n_{\mathbf{k}}\rangle &= a^{\dagger}(\mathbf{k})\,a(\mathbf{k})\,a^{\dagger}(\mathbf{k})\,|n_{\mathbf{k}}\rangle &(5)\\
&= a^{\dagger}(\mathbf{k})\left[1+a^{\dagger}(\mathbf{k})\,a(\mathbf{k})\right]|n_{\mathbf{k}}\rangle &(6)\\
&= a^{\dagger}(\mathbf{k})\left[1+N_a(\mathbf{k})\right]|n_{\mathbf{k}}\rangle &(7)\\
&= a^{\dagger}(\mathbf{k})\left[1+n_{\mathbf{k}}\right]|n_{\mathbf{k}}\rangle &(8)\\
&= \left[1+n_{\mathbf{k}}\right]a^{\dagger}(\mathbf{k})\,|n_{\mathbf{k}}\rangle &(9)
\end{aligned}$$

Thus the state $a^{\dagger}\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle$ is an eigenstate of $N_{a}\left(\mathbf{k}\right)$ with eigenvalue $1+n_{\mathbf{k}}$ so the operator $a^{\dagger}\left(\mathbf{k}\right)$ has created an extra $\omega_{\mathbf{k}}$ particle.

The procedure for $a\left(\mathbf{k}\right)$ is similar:

$$\begin{aligned}
N_{a}\left(\mathbf{k}\right)a\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle &= a^{\dagger}\left(\mathbf{k}\right)a\left(\mathbf{k}\right)a\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle && (10)\\
&= \left[a\left(\mathbf{k}\right)a^{\dagger}\left(\mathbf{k}\right)-1\right]a\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle && (11)\\
&= \left[a\left(\mathbf{k}\right)N_{a}\left(\mathbf{k}\right)-a\left(\mathbf{k}\right)\right]\left|n_{\mathbf{k}}\right\rangle && (12)\\
&= a\left(\mathbf{k}\right)\left[n_{\mathbf{k}}-1\right]\left|n_{\mathbf{k}}\right\rangle && (13)\\
&= \left[n_{\mathbf{k}}-1\right]a\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle && (14)
\end{aligned}$$

Thus the state $a\left(\mathbf{k}\right)\left|n_{\mathbf{k}}\right\rangle$ is an eigenstate of $N_{a}\left(\mathbf{k}\right)$ with eigenvalue $n_{\mathbf{k}}-1$ so the operator $a\left(\mathbf{k}\right)$ has destroyed a $\omega_{\mathbf{k}}$ particle.

We can represent a state containing $n_{\mathbf{k}}$ b type particles (antiparticles) by $\left|\bar{n}_{\mathbf{k}}\right\rangle$ (a bar over a symbol means it refers to antiparticles). Then

$$N_{b}\left(\mathbf{k}\right)\left|\bar{n}_{\mathbf{k}}\right\rangle = \bar{n}_{\mathbf{k}}\left|\bar{n}_{\mathbf{k}}\right\rangle \quad (15)$$

The two derivations above are exactly the same for antiparticles, if we replace all a operators by b operators, N_{a} by N_{b} and $n_{\mathbf{k}}$ by $\bar{n}_{\mathbf{k}}$. Thus $b^{\dagger}\left(\mathbf{k}\right)$ creates a single antiparticle with energy $\omega_{\mathbf{k}}$ and $b\left(\mathbf{k}\right)$ destroys one antiparticle with that energy.

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